Integration without tears

Bengt Ringnér, Matematikcentrum, Lund

Abstract: Rather than working through tedious measure theory before reaching the central results, Daniell (1918) started with any integral, I, having the properties of the Riemann one plus

- $g_n \searrow 0$ implies $I(g_n) \to 0$.

Instead of continuing with upper and lower integrals, we follow Stone (1948) and define a seminorm by

$$\|f\| := \inf\Big\{\sum_k I(g_k) : \sum_k g_k \geq |f|, \quad 0 \leq g_k \in E\Big\}$$

where E denotes the domain of I. Stone defined the integrable functions as those in L, the closure of E. It follows readily from the $\bullet$ that

$$|I(g)| \leq I(|g|) = \|g\|$$

for g in E, that is, I is a (uniformly) continuous linear functional, whence it can be extended uniquely to L. This idea seems due to Bourbaki. The extension is denoted by $\int$. To make the picture complete, using the reverse triangle inequality on $\|f - g_n\| \to 0$ gives

$$\|f\| = \int |f|$$

for all integrable f.

Finally, the "Super Beppo Levi theorem" follows directly: If $S(x) = \sum_n f_n(x)$ when $\sum_n |f_n(x)| < \infty$ and $\sum \int |f_n| < \infty$, then S is integrable, and

$$\int S = \sum_n \int f_n.$$