Lecture 4
Diversity Techniques

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Outline

• Diversity (Chapter 7 in Goldsmith’s Book)
  • Data Transmission using Multiple Carriers
  • Multicarrier Modulation with Overlapping Subchannels
  • Mitigation of Subcarrier Fading
  • Discrete Implementation of Multicarrier
  • Challenges in Multicarrier Systems
Introduction

• One of the most powerful techniques to mitigate the effects of fading is to use diversity-combining of independently fading signal paths.

• Diversity-combining uses the fact that independent signal paths have a low probability of experiencing deep fades simultaneously.

• The idea behind diversity is to send the same data over independent fading paths.

• These independent paths are combined in some way such that the fading of the resultant signal is reduced.

• Diversity techniques that mitigate the effect of multipath fading are called microdiversity.

• Diversity to mitigate the effects of shadowing from buildings and objects is called macrodiversity.
Realization of Independent Fading Paths

• There are many ways of achieving independent fading paths in a wireless system. One method is to use multiple transmit or receive antennas, also called an antenna array, where the elements of the array are separated in distance. This type of diversity is referred to as space diversity.
• Note that with receiver space diversity, independent fading paths are realized without an increase in transmit signal power or bandwidth.
• Coherent combining of the diversity signals leads to an increase in SNR at the receiver over the SNR that would be obtained with just a single receive antenna.
• To obtain independent paths through transmitter space diversity, the transmit power must be divided among multiple antennas.
• Space diversity also requires that the separation between antennas be such that the fading amplitudes corresponding to each antenna are approximately independent.
• A second method of achieving diversity is by using either two transmit antennas or two receive antennas with different polarization.
Realization of Independent Fading Paths

• There are two disadvantages of polarization diversity.
  • First, you can have **at most two diversity branches**, corresponding to the two types of polarization.
  • The second disadvantage is that polarization diversity loses effectively half the power (3 dB) since the transmit or receive power is divided between the two differently polarized antennas.

• **Directional antennas** provide angle, or directional, diversity by restricting the receive antenna beamwidth to a given angle. **Smart antennas are antenna arrays with adjustable phase at each antenna element:** such arrays form directional antennas that can be steered to the incoming angle of the strongest multipath component.

• **Frequency diversity** is achieved by transmitting the same narrowband signal at different carrier frequencies, where the carriers are separated by the coherence bandwidth of the channel.

• **Time diversity** is achieved by transmitting the same signal at different times, where the time difference is greater than the channel coherence time (the inverse of the channel Doppler spread).
Receiver Diversity

• Time diversity does not require increased transmit power, but it does decrease the data rate since data is repeated in the diversity time slots rather than sending new data in these time slots.

• Clearly time diversity can’t be used for stationary applications, since the channel coherence time is infinite and thus fading is highly correlated over time.

• In receiver diversity the independent fading paths associated with multiple receive antennas are combined to obtain a resultant signal that is then passed through a standard demodulator.

• Most combining techniques are linear: the output of the combiner is just a weighted sum of the different fading paths or branches, as shown in Figure 7.1 for M-branch diversity.
Receiver Diversity

Figure 7.1: Linear Combiner.
Receiver Diversity

- Combining more than one branch signal requires **co-phasing**, where the phase $\theta_i$ of the $i$th branch is removed through the multiplication by $\alpha_i = a_i e^{-j\theta_i}$ for some real-valued $a_i$.
- This phase removal requires coherent detection of each branch to determine its phase $\theta_i$.
- **Without co-phasing**, the branch signals would not add up coherently in the combiner, so the resulting output could still exhibit significant fading.
- Combining is typically performed post-detection, since the branch signal power and/or phase is required to determine the appropriate $\alpha_i$ value.
- The main purpose of diversity is to coherently combine the independent fading paths so that the effects of fading are mitigated.
- The signal output from the combiner equals the original transmitted signal $s(t)$ multiplied by a random complex amplitude term $\alpha_{\Sigma} = \sum_i a_i r_i$
- This complex amplitude term results in a random SNR $\gamma_{\Sigma}$ at the combiner output, where the distribution of $\gamma_{\Sigma}$ is a function of the number of diversity paths, the fading distribution on each path, and the combining technique.
Receiver Diversity

- There are two types of performance gain associated with receiver space diversity: **array gain** and **diversity gain**.
- **The array gain** results from coherent combining of multiple receive signals. Even in the absence of fading, this can lead to an increase in average received SNR.
- For example, suppose there is no fading so that \( r_i = \sqrt{E_s} \) for \( E_s \) the energy per symbol of the transmitted signal. Assume identical noise PSD \( N_0 \) on each branch and pulse shaping such that \( BT_s = 1 \).
- Then each branch has the same SNR \( \gamma_i = E_s/N_0 \).
- Let us set \( a_i = r_i/\sqrt{N_0} \) : we will see later that these weights are optimal for **maximal-ratio combining (MRC)** in fading. Then the received SNR is

\[
\gamma_\Sigma = \frac{\left( \sum_{i=1}^{M} a_i r_i \right)^2}{N_0 \sum_{i=1}^{M} a_i^2} = \frac{\left( \sum_{i=1}^{M} E_s \right)^2}{N_0 \sum_{i=1}^{M} E_s} = \frac{M E_s}{N_0}. \tag{7.1}
\]

- Thus, in the absence of fading, with appropriate weighting there is an \( M \)-fold increase in SNR due to the coherent combining of the \( M \) signals received from the different antennas.
Receiver Diversity

- This SNR increase in the absence of fading is referred to as the **array gain**.
- More precisely, array gain $A_g$ is defined as the increase in averaged combined SNR $\bar{\gamma}_\Sigma$ over the average branch SNR $\bar{\gamma}$:

$$A_g = \frac{\bar{\gamma}_\Sigma}{\bar{\gamma}}.$$  

- The array gain allows a system with multiple transmit or receive antennas in a fading channel to achieve better performance than a system without diversity in an AWGN channel with the same average SNR.
- In fading the combining of multiple independent fading paths leads to a more favorable distribution for $\gamma_\Sigma$ than would be the case with just a single path.
- (Average) Probability of Symbol Error:

$$\overline{P}_s = \int_0^\infty P_s(\gamma)p_{\gamma_\Sigma}(\gamma)d\gamma,$$  \hspace{1cm} (7.2)

where $P_s(\gamma)$ is the probability of symbol error for demodulation of $s(t)$ in AWGN with SNR $\gamma_\Sigma$.  

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Receiver Diversity : SC

• For some diversity systems their average probability of error can be expressed in the form $P_s = c\eta^{-M}$ where $c$ is a constant that depends on the specific modulation and coding, $\eta$ is the average received $SNR$ per branch, and $M$ is called the diversity order of the system.

• The diversity order indicates how the slope of the average probability of error as a function of average $SNR$ changes with diversity.

• The maximum diversity order of a system with $M$ antennas is $M$, and when the diversity order equals $M$ the system is said to achieve FULL diversity order.

• **Selection Combining:** In selection combining (SC), the combiner outputs the signal on the branch with the highest $SNR$ $r_i^2/N_i$.

• Since only one branch is used at a time, SC often requires just one receiver that is switched into the active antenna branch.

• However, a dedicated receiver on each antenna branch may be needed for systems that transmit continuously in order to simultaneously and continuously monitor $SNR$ on each branch.
Receiver Diversity: SC

- For $M$ branch diversity, the CDF of $\gamma_\Sigma$ is given by

$$P_{\gamma_\Sigma}(\gamma) = p(\gamma_\Sigma < \gamma) = p(\max[\gamma_1, \gamma_2, \ldots, \gamma_M] < \gamma) = \prod_{i=1}^{M} p(\gamma_i < \gamma). \quad (7.4)$$

- Assume that we have $M$ branches with uncorrelated Rayleigh fading amplitudes $r_i$. The instantaneous SNR on the $i$th branch is therefore given by $\gamma_i = r_i^2 / N$.

- Defining the average SNR on the $i$th branch as $\overline{\gamma_i} = E[\gamma_i]$, the SNR distribution will be exponential:

$$p(\gamma_i) = \frac{1}{\overline{\gamma_i}} e^{-\gamma_i / \overline{\gamma_i}}. \quad (7.5)$$

- The outage probability for a target $\gamma_0$ on the $i$th branch in Rayleigh fading is

$$P_{out}(\gamma_0) = 1 - e^{-\gamma_0 / \overline{\gamma_i}}. \quad (7.6)$$
Receiver Diversity : SC

• The outage probability of the selection-combiner for the target $\gamma_0$ is then

$$P_{out}(\gamma_0) = \prod_{i=1}^{M} p(\gamma_i < \gamma_0) = \prod_{i=1}^{M} \left[ 1 - e^{-\gamma_0/\bar{\gamma}_i} \right].$$  \hspace{1cm} (7.7)

• If the average SNR for all of the branches are the same ($\bar{\gamma}_i = \bar{\gamma}$ for all $i$), then this reduces to

$$P_{out}(\gamma_0) = p(\gamma_\Sigma < \gamma_0) = \left[ 1 - e^{-\gamma_0/\bar{\gamma}} \right]^M.$$  \hspace{1cm} (7.8)

Differentiating $P_{out}(\gamma_0)$ relative to $\gamma_0$ yields the pdf for $\gamma_\Sigma$:

$$p_{\gamma_\Sigma}(\gamma) = \frac{M}{\bar{\gamma}} \left[ 1 - e^{-\gamma/\bar{\gamma}} \right]^{M-1} e^{-\gamma/\bar{\gamma}}.$$  \hspace{1cm} (7.9)

• So we see that the average SNR of the combiner output in i.i.d. Rayleigh fading is
Receiver Diversity : SC

\[ \bar{\gamma}_\Sigma = \int_0^\infty \gamma p_{\gamma \Sigma} (\gamma) d\gamma \]
\[ = \int_0^\infty \frac{\gamma M}{\bar{\gamma}} \left[ 1 - e^{-\gamma/\bar{\gamma}} \right]^{M-1} e^{-\gamma/\bar{\gamma}} d\gamma \]
\[ = \bar{\gamma} \sum_{i=1}^{M} \frac{1}{i}. \]  

(7.10)

Thus, the average SNR gain increases with \( M \), but not linearly.

- The biggest gain is obtained by going from no diversity to two-branch diversity. Increasing the number of diversity branches from two to three will give much less gain than going from one to two, and in general increasing \( M \) yields diminishing returns in terms of the SNR gain.
- This trend is also illustrated in Figure 7.2, which shows \( P_{out} \) versus \( \bar{\gamma}/\gamma_0 \) for different \( M \) in i.i.d. Rayleigh fading.
Figure 7.2: Outage Probability of Selection Combining in Rayleigh Fading.
Receiver Diversity : MRC

- In maximal ratio combining (MRC) the output is a weighted sum of all branches, so the \( \alpha_i \) in Figure 7.1 are all nonzero.
- The envelope of the combiner output will be \( r = \sum_{i=1}^{M} a_i r_i \).
- Assuming the same noise PSD \( N_0 \) in each branch yields a total noise PSD \( N_{tot} \) at the combiner output of \( N_{tot} = \sum_{i=1}^{M} a_i^2 N_0 \). Thus, the output SNR of the combiner is

\[
\gamma_{\Sigma} = \frac{r^2}{N_{tot}} = \frac{1}{N_0} \frac{\left( \sum_{i=1}^{M} a_i r_i \right)^2}{\sum_{i=1}^{M} a_i^2}.
\]  
(7.17)

- The goal is to choose the \( \alpha_i \)'s to maximize \( \gamma_{\Sigma} \). Intuitively, branches with a high SNR should be weighted more than branches with a low SNR, so the weights \( a_i^2 \) should be proportional to the branch SNRs \( r_i^2 / N_0 \).
- Solving for the optimal weights yields \( a_i^2 = r_i^2 / N_0 \), and the resulting combiner SNR becomes \( \gamma_{\Sigma} = \sum_{i=1}^{M} r_i^2 / N_0 = \sum_{i=1}^{M} \gamma_i \cdot \)
- Thus, the SNR of the combiner output is the sum of SNRs on each branch.
The average combiner SNR increases linearly with the number of diversity branches $M$, in contrast to the diminishing returns associated with the average combiner SNR in SC.

Note that even with Rayleigh fading on all branches, the distribution of the combiner output SNR is no longer Rayleigh.

Assuming i.i.d. Rayleigh fading on each branch with equal average branch SNR $ar{\gamma}$, the distribution of $\gamma_{\Sigma}$ is chi-squared with $2M$ degrees of freedom, expected value $\bar{\gamma}_{\Sigma} = M\bar{\gamma}$, and variance $2M\bar{\gamma}$:

$$p_{\gamma_{\Sigma}}(\gamma) = \frac{\gamma^{M-1}e^{-\gamma/\bar{\gamma}}}{\bar{\gamma}^M(M-1)!}, \quad \gamma \geq 0.$$  

The corresponding outage probability for a given threshold $\gamma_0$ is given by

$$P_{out} = p(\gamma_{\Sigma} < \gamma_0) = \int_0^{\gamma_0} p_{\gamma_{\Sigma}}(\gamma) d\gamma = 1 - e^{-\gamma_0/\bar{\gamma}} \sum_{k=1}^{M} \frac{(\gamma_0/\bar{\gamma})^{k-1}}{(k-1)!}.$$  

Figure 7.5 plots $P_{out}$ for MRC indexed by the number of diversity branches.
Receiver Diversity: MRC

Figure 7.5: $P_{out}$ for MRC with i.i.d. Rayleigh fading.
Lecture 4: Diversity

Receiver Diversity : MRC

• For BPSK modulation with i.i.d Rayleigh fading, it can be shown that

\[
\overline{P_b} = \int_0^\infty Q(\sqrt{2\gamma}) p_{\gamma\Sigma}(\gamma) d\gamma = \left(\frac{1 - \Gamma}{2}\right)^M \sum_{m=0}^{M-1} \binom{M - 1 + m}{m} \left(\frac{1 + \Gamma}{2}\right)^m,
\]

where \(\Gamma = \sqrt{\frac{\gamma}{1 + \gamma}}\). This equation is plotted in Figure 7.6.

• We can obtain a simple upper bound on the average probability of error by applying the Chernoff bound \(Q(x) \leq e^{-x^2/2}\) to the \(Q\) function.

• Recall that for static channel gains with MRC, we can approximate the probability of error as

\[
P_s = \alpha_M Q(\sqrt{\beta_M \gamma\Sigma}) \leq \alpha_M e^{-\beta_M \gamma\Sigma/2} = \alpha_M e^{-\beta_M (\gamma_1 + \ldots + \gamma_M)/2}.
\]

• Integrating over the chi-squared distribution for \(\gamma\Sigma\) yields

\[
\overline{P_s} \leq \alpha_M \prod_{i=1}^{M} \frac{1}{1 + \beta_M \overline{\gamma}_i/2}. \quad \gamma_i\text{'s i.i.d., large}
\]

\[
\overline{P_s} \approx \alpha_M \left(\frac{\beta_M \overline{\gamma}}{2}\right)^{-M}.
\]
Receiver Diversity: MRC

Figure 7.6: $P_b$ for MRC with i.i.d. Rayleigh fading.
Receiver Diversity: EGC

- MRC requires knowledge of the time-varying SNR on each branch, which can be very difficult to measure.
- A simpler technique is equal-gain combining (EGC), which co-phases the signals on each branch and then combines them with equal weighting, \( \alpha_i = e^{-\theta_i} \).
- The SNR of the combiner output, assuming equal noise PSD \( N_0 \) in each branch, is then given by

\[
\gamma_\Sigma = \frac{1}{N_0 M} \left( \sum_{i=1}^{M} r_i \right)^2.
\]

For i.i.d. Rayleigh fading and two-branch diversity and average branch SNR \( \overline{\gamma} \), an expression for the CDF in terms of the \( Q \) function can be derived as

\[
P_{\gamma_\Sigma}(\gamma) = 1 - e^{-2\gamma/\overline{\gamma}} \sqrt{\frac{\pi\overline{\gamma}}{\gamma}} e^{-\gamma/\overline{\gamma}} \left( 1 - 2Q \left( \sqrt{2\gamma/\overline{\gamma}} \right) \right).
\]
Receiver Diversity : EGC

- The resulting outage probability is given by

\[ P_{out}(\gamma_0) = 1 - e^{-2\gamma_R} - \sqrt{\pi \gamma_R} e^{-\gamma_R} \left( 1 - 2Q\left(\sqrt{2\gamma_R}\right) \right), \]

where \( \gamma_R = \gamma_0 / \bar{\gamma} \). The pdf of \( \gamma \) is given by

\[ p_{\gamma\Sigma}(\gamma) = \frac{1}{\bar{\gamma}} e^{-2\gamma / \bar{\gamma}} + \sqrt{\pi} e^{-\gamma / \bar{\gamma}} \left( \frac{1}{\sqrt{4\gamma \bar{\gamma}}} - \frac{1}{\bar{\gamma}} \sqrt{\gamma / \bar{\gamma}} \right) \left( 1 - 2Q(\sqrt{2\gamma / \bar{\gamma}}) \right). \]

- For BPSK, the average probability of bit error is

\[ \overline{P}_b = \int_0^\infty Q(\sqrt{2\gamma}) p_{\gamma\Sigma}(\gamma) d\gamma = .5 \left( 1 - \sqrt{1 - \left( \frac{1}{1 + \bar{\gamma}} \right)^2} \right). \] (7.28)
Transmitter Diversity

- In transmit diversity there are multiple transmit antennas with the transmit power divided among these antennas.
- Transmit diversity design depends on whether or not the complex channel gain is known at the transmitter or not.
- **Channel Known at Transmitter**: Consider a transmit diversity system with $M$ transmit antennas and one receive antenna. We assume the path gain associated with the $i$th antenna given by $r_i e^{j\theta_i}$ is known at the transmitter.
- Let $s(t)$ denote the transmitted signal and the weighted signals transmitted over all antennas are added “in the air”, which leads to a received signal given by

$$r(t) = \sum_{i=1}^{M} a_i r_i s(t).$$

- Suppose we wish to set the branch weights to maximize received SNR. Using a similar analysis as in receiver MRC diversity, we see that the weights $a_i$ that achieve the maximum SNR are given by
Transmitter Diversity

\[ a_i = \frac{r_i}{\sqrt{\sum_{i=1}^{M} r_i^2}}, \]

and the resulting SNR is

\[ \gamma\Sigma = \frac{E_s}{N_0} \sum_{i=1}^{M} r_i^2 = \sum_{i=1}^{M} \gamma_i, \quad (7.31) \]

- For \( \gamma_i = r_i^2 E_s / N_0 \) equal to the branch SNR between the \( i \)th transmit antenna and the receive antenna.
- Thus we see that transmit diversity when the channel gains are known at the transmitter is very similar to receiver diversity with MRC: the received SNR is the sum of SNRs on each of the individual branches.

- **Channel Unknown at Transmitter - The Alamouti Scheme**
- We now consider that the transmitter no longer knows the channel gains, so there is no CSIT.
Transmitter Diversity

• In this case it is not obvious how to obtain diversity gain. Consider, for example, a naive strategy whereby for a two-antenna system we divide the transmit energy equally between the two antennas.
• Thus, the transmit signal on antenna \( i \) will be \( s_i(t) = \sqrt{.5}s(t) \) for \( s(t) \) the transmit signal with energy per symbol \( E_s \).
• The received signal is then

\[
r(t) = \sqrt{.5}(h_1 + h_2)s(t).
\]

• Note that \( h_1 + h_2 \) is the sum of two complex Gaussian random variables, and is thus a complex Gaussian as well with mean equal to the sum of means (zero) and variance equal to the sum of variances 2.
• Thus \( \sqrt{.5}(h_1 + h_2) \) is a complex Gaussian random variable with mean zero and variance one, so the received signal has the same distribution as if we had just used one antenna with the full energy per symbol.
• In other words, we have obtained no performance advantage from the two antennas, since we could not divide our energy intelligently between them or obtain coherent combining through co-phasing.
Transmitter Diversity

• Transmit diversity gain can be obtained even in the absence of channel information with an appropriate scheme to exploit the antennas. A particularly simple and prevalent scheme for this diversity that combines both space and time diversity was developed by Alamouti.

• Alamouti’s scheme is designed for a digital communication system with two-antenna transmit diversity. The scheme works over two symbol periods where it is assumed that the channel gain is constant over this time.

• Over the first symbol period two different symbols \( s_1 \) and \( s_2 \) each with energy \( E_s/2 \) are transmitted simultaneously from antennas 1 and 2, respectively.

• Over the next symbol period symbol \( -s_2^* \) is transmitted from antenna 1 and symbol \( s_1^* \) is transmitted from antenna 2, each with symbol energy \( E_s/2 \).

• \( y = [y_1 y_2^*]^T \) uses these sequentially received symbols to form the vector given by:

\[
y = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix} = H_A s + n,
\]
Transmitter Diversity

where \( s = [s_1 s_2]^T \), \( n = [n_1 n_2]^T \), and \( H_A = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \).

- Let us define the new vector \( z = H_A^H y \). The structure of \( H_A \) implies that

\[
H_A^H H_A = (|h_1|^2 + |h_2|^2)I_2,
\]

is diagonal, and thus

\[
z = [z_1 \ z_2]^T = (|h_1|^2 + |h_2|^2)I_2 s + \tilde{n},
\]

where \( \tilde{n} = H_A^H n \) is a complex Gaussian noise vector with mean zero and covariance matrix

\[
E[\tilde{n}\tilde{n}^*] = (|h_1|^2 + |h_2|^2)N I_2.
\]

- The diagonal nature of \( z \) effectively decouples the two symbol transmissions, so that each component of \( z \) corresponds to one of the transmitted symbols:

\[
z_i = (|h_1|^2 + |h_2|^2)s_i + \tilde{n}_i, \quad i = 1, 2.
\]
Transmitter Diversity

• The received SNR thus corresponds to the SNR for \( z_i \) given by

\[
\gamma_i = \frac{(|h_1^2| + |h_2^2|) E_s}{2N_0},
\]

where the factor of 2 comes from the fact that \( s_i \) is transmitted using half the total symbol energy \( E_s \).

• The receive SNR is thus equal to the sum of SNRs on each branch, identical to the case of transmit diversity with MRC assuming that the channel gains are known at the transmitter.

• Thus, the Alamouti scheme achieves a diversity order of 2, the maximum possible for a two-antenna transmit system, despite the fact that channel knowledge is not available at the transmitter.

• However, it only achieves an array gain of 1, whereas MRC can achieve an array gain and a diversity gain of 2.