Rectangular Spiral Galaxies are Still Hard*

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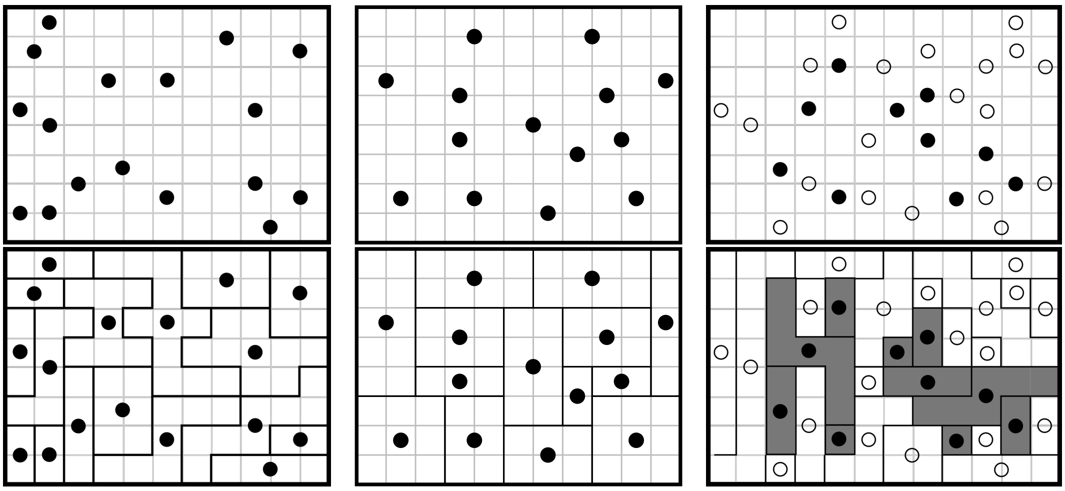
Abstract

Spiral Galaxies is a pencil-and-paper puzzle played on a grid of unit squares with a given set of points, referred to as *centers*. The grid needs to be partitioned into polyominoes such that each polyomino contains exactly one center and is 180° rotationally symmetric about its center. We show that the puzzle is NP-complete even if the polyominoes are restricted to be (a) rectangles of arbitrary size or (b) $1{\times}1$, $1{\times}3$, and $3{\times}1$ rectangles. The proof for the latter variant also implies that finding a non-crossing matching in modified grid graphs where edges connect vertices of distance 2 is NP-complete. Moreover, we prove that minimizing the number of centers, such that there exist a set of Spiral Galaxies that exactly cover a given shape, is NP-complete.

1 Introduction



(a) (b) (c)

Figure 1 Spiral Galaxies puzzles in several styles; solutions in the bottom row. (a) Classic Spiral Galaxies puzzle. (b) Rectangular Galaxies puzzle. (c) Spiral Galaxies puzzle with black and white centers such that the polyominoes containing the black centers in the solution yield a picture.

Spiral Galaxies is a pencil-and-paper puzzle published by Nikoli in 2001 [5] under the name "Tentai Show" [6]. It is played on a grid of unit squares with given *centers*: points

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that are located at grid vertices, square centers or edge midpoints. The goal is to decompose the grid into polyominoes, such that each polyomino contains exactly one center and is 180° rotationally symmetric about its center; see Figure 1 (a). The solution for a Spiral Galaxies puzzle may not be unique, but generally puzzles are designed to have a unique solution.

Friedman [4] showed Spiral Galaxies to be NP-complete for general polyomino shapes, and Fertin et al. [3] showed NP-hardness for Spiral Galaxies of size at most seven. In this paper we consider Rectangular Galaxies; a variant of Spiral Galaxies in which all polyominoes are required to be rectangles; see Figure 1 (b). We consider the general case and a special case in which we allow only 1×1, 1×3, and 3×1 rectangles. We show that both puzzle variants are (still) NP-complete. The proof for the latter puzzle also implies that finding a non-crossing matching in squared grid graphs (that is, modified grid graphs where edges connect vertices at distance 2) is NP-complete.

In another variant of the puzzle, a subset of the centers is colored black, and the polyominoes that contain these centers yield a picture as a solution [6]; see Figure 1 (c). Logic puzzles which yield pictures when solved are a popular genre; one famous example is the nonogram [7]. Such puzzles can also be used to construct fonts [1]. Constructing an interesting puzzle such that its solution is a given target shape is non-trivial: while a valid puzzle trivially exists, by simply placing a center in every grid cell, the resulting puzzle is clearly not interesting. We are, hence, also interested in finding the minimum number of centers, such that there exist spiral galaxies that exactly cover a given shape. We also prove this problem to be NP-complete.

1.1 Notation and Preliminaries

The game is played on an $m \times n$ grid. Centers are placed on the grid (on square centers, edge midpoints, or grid vertices), either all of them have the same color, or a subset of the centers may be colored black. To solve each puzzle, the solver is required to partition the board with polyominoes, such that each polyomino contains a single center and is 180° rotationally symmetric about its center. The grid and the centers form a Spiral Galaxies board B.

For our reductions, we use the PLANAR 1-IN-3-SAT PROBLEM, a well known NP-complete and $\#P$-complete problem [2].

2 Rectangular Galaxies

In this section, we show that the problem of solving Sprial Galaxies boards is NP-complete when all galaxies are restricted to be rectangles.

▶ **Theorem 1.** *Determining if a Spiral Galaxies board is solvable with only rectangular galaxies is NP-complete.*

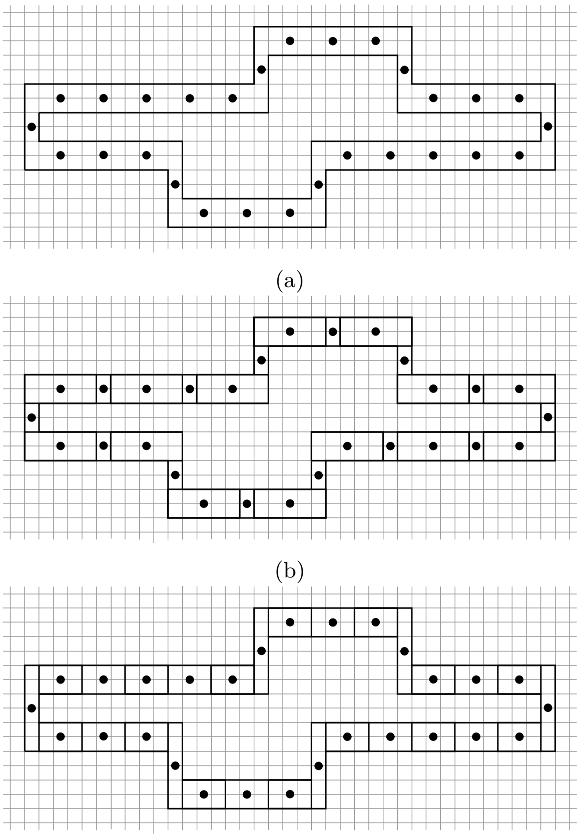
Proof. We give a reduction from PLANAR 1-IN-3-SAT. Given an instance F of PLANAR 1-IN-3-SAT with incidence graph G, we show how to turn a rectilinear planar embedding of G into a Spiral Galaxies board B such that a solution to B yields a solution to F.

We begin by constructing a representation of variables, we then show how to propagate and bend the variable values using 'wires'/'corridors' and combine the wires to form clauses.

The grid regions not incorporated into our gadgets will not affect the gadget solutions, as for any open region we use a **face gadget**, shown in Figure 2, to force the region to contain single square galaxies, which disallow any other Spiral Galaxies to occupy these grid squares. Therefore, there can be no interference between our gadgets and the grid regions surrounding

(a) (b)

Figure 2 Face gadget to fill in the space inbetween all other gadgets.



(a)

(b)

(c)

Figure 3 (a) Variable loop with two possible states (b) and (c) corresponding to a truth assignment of "true" and "false" of the corresponding variable.

them. Note that our face gadget forces us to not use open regions of width 1, as these would not enforce single square galaxies.

We construct **variable loops** as shown in Figure 3. They are constructed with centers located at edge centers within distance of 3 of each other, the loop is closed by additional centers at edge center points. Each variable loop has two possible solutions, corresponding to setting the variable as "true" or "false", and the rectangle placement is completely determined by the variable assignment.

From each variable loop, we can propagate the variable value, creating a **corridor gadget**, shown in Figure 4. The corridor gadgets are shown in blue and have a center distance of 5 on edge centers. Again, only two truth assignments are possible, each correspond to setting the variable to "true" (Figure 4(b)) or "false" (Figure 4(c)). If the corridor does not pick up the correct signal from the variable loop, the variable loop cannot be solved with rectangular

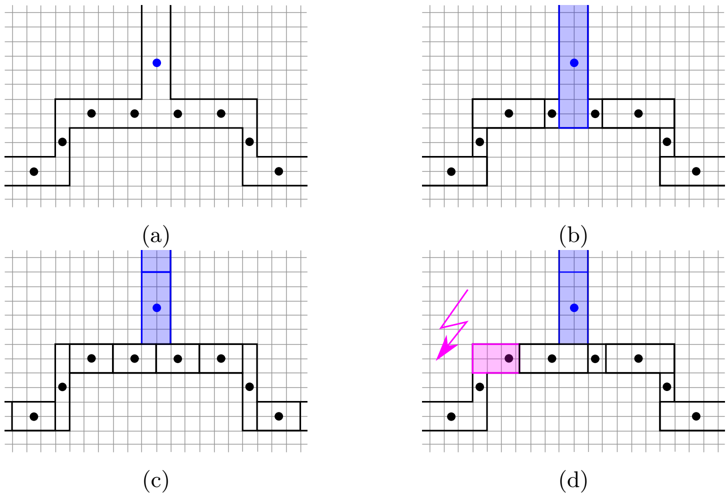


Figure 4 (a) Variable loops (black centers) and connecting variable corridors (blue centers). (b)/(c) Two possible truth assignments of the connecting corridor depending on the truth assignment in the variable loop. (d) If the corridor does not pick up the correct signal from the variable loop the variable loop cannot be completed.

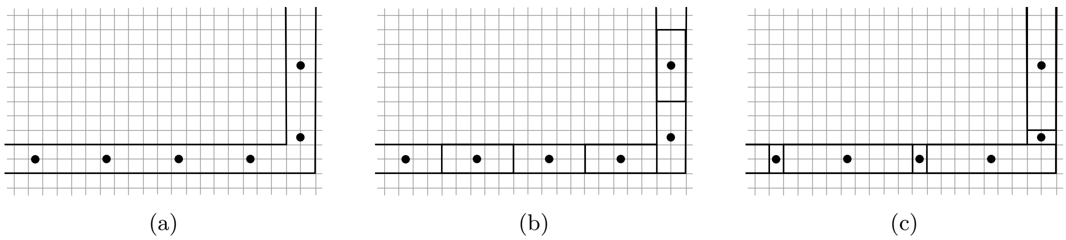


Figure 5 (a) Variable corridors with bend, (b)/(c) two possible covers with rectangular galaxies.

Spiral Galaxies, see, e.g., Figure 4(d). We show the **corridor bend** gadget in Figure 5. Moreover, we use a **corridor shift** gadget, shown in Figure 6, which allows us to shift the location of the corridor by any number larger than 3. These are used to connect to the clause gadgets in the appropriate places.

To combine the corridor gadgets into clauses, we use the **clause gadget** shown in Figure 7. At least one of the three variables' truth assignments is forced to be true (see Figure 7(b)-(d)), more than one cannot be set to true (see Figure 7(e)-(h)), hence, it forces exactly 1 true assignment, giving a solution to the instance F.

A solution to an $m \times n$ Spiral Galaxies board can be verified in polynomial time. ◀

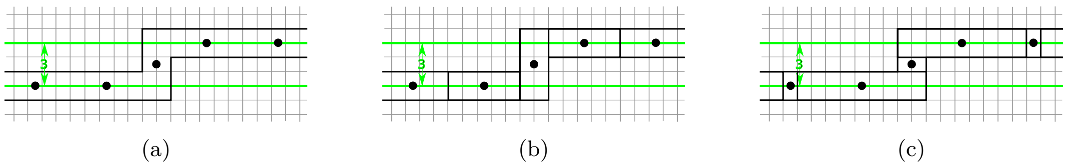


Figure 6 (a) Variable corridors can be shifted by any number larger than 3 (here shown for a shift of 3 as indicated in green). (b) and (c) depict the different variable assignments.

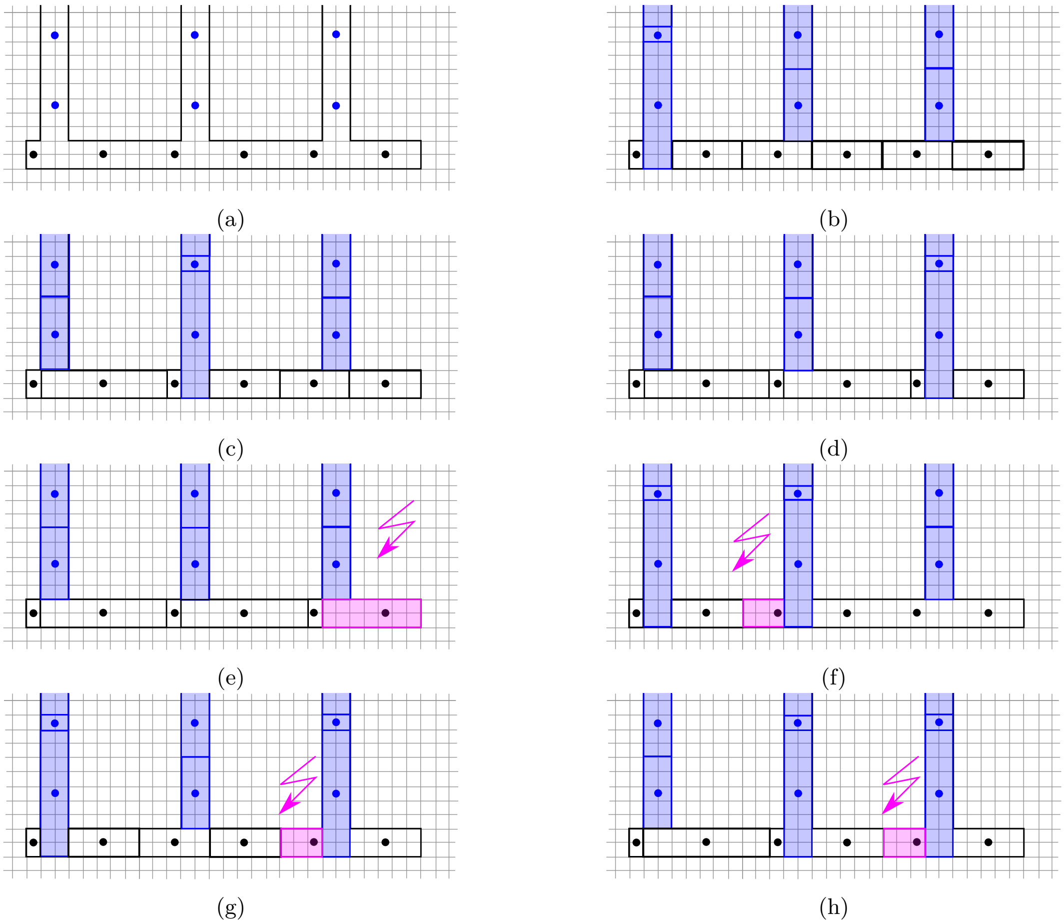


Figure 7 (a) Clause gadget shown in black and the connecting variable corridors shown in blue. (b)-(d) The clause gadget can be filled in with rectangle galaxies iff exactly one of the three connecting variables has a truth assignment that fulfills the clause. The clause cannot be completed if all variables do not fulfill the clause (e), or if more than one variable fulfills the clause (f)-(h).

3 Spiral Galaxies with 1×1, 1×3, and 3×1 Rectangles

In this section, we show that the problem of solving Spiral Galaxies boards is NP-complete, even when only $1 \times 1, 1 \times 3$ and 3×1 galaxies are allowed, and that the problem of counting the number of solutions to a Spiral Galaxies board with these galaxy types is $\#P$-complete and ASP-complete, see [8] Chapter 28 and [9, 10] for definitions of $\#P$-completeness and ASP-completeness, respectively. For our reductions, we—again—use the PLANAR 1-IN-3-SAT PROBLEM.

▶ **Theorem 2.** *Determining if a Spiral Galaxies board is solvable with only* $1 \times 1, 1 \times 3$ *and* 3×1 *galaxies is NP-complete and counting the number of solutions is #P-complete and ASP-complete.*

Proof. The proof is by reduction from PLANAR 1-IN-3-SAT. Given an instance F of planar 1-in-3-SAT with incidence graph G, we show how to turn a rectilinear planar embedding of G into a Spiral Galaxies board B such that a solution to B yields a solution to F, thereby showing NP-completeness. Furthermore, there will be a one-to-one correspondence between solutions of B and solutions of F, showing #P-completeness and ASP-completeness.

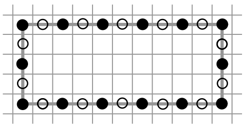
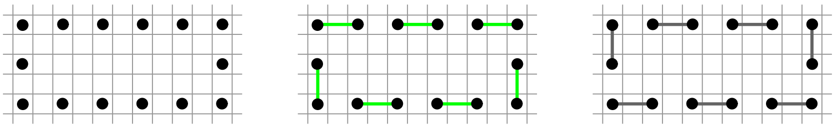


Figure 8 We place centers (black circles) on the middle point of potential edges (light gray) between any pair of black disks. In the remainder of this Section, we show only the black disks, but not the centers.



(a) (b) (c)

Figure 9 (a) Variable loop with two possible states (b) and (c) corresponding to a truth assignment of "true" and "false" of the corresponding variable.

In this proof, disks in the figures will not show centers for Spiral Galaxies. Disks with distance 2 can be connected by an edge, centers will be located at the middle of each potential edge, see Figure 8. Hence, any 1×3 or 3×1 galaxy will cover both disks, denoted by an edge between these two disks; any 1×1 galaxy will not extend over the disks, and is shown by a non-existing edge between the disks.

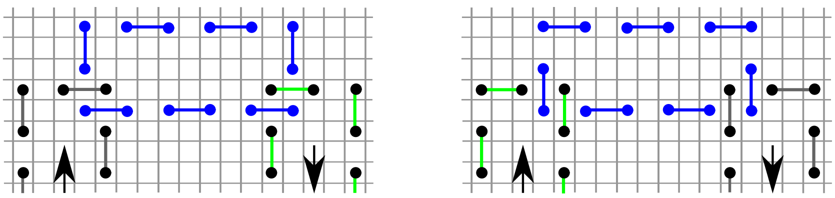
We start with constructing variable gadgets that represent variables and their negations, we show how we can negate the truth assignment of a variable, and construct clause gadgets in which we combine the variable loops. Throughout the discussion, the constructed gadgets have a single solution for any given variable assignment, which will make this reduction parsimonious.

The board region not incorporated into our gadgets will be filled with centers at square centers which will force the region to be filled with unit square galaxies, and will disallow any other type of galaxies. Thus, there is no interference between our gadgets and the board area surrouding them. Again, the face gadget forces us to not use open regions of width 1, as these would not enforce single square galaxies.

The **variable loop**, shown in Figure 11, has two possible solutions (shown in Figure 11(b) and (c)), each corresponding to one truth assignment for the variable ("true" and "false"). We extend the size of the variable loop to connect to other gadgets—to build **corridors**.

Negating a variable corresponds to inserting a **negation gadget** into the corridor, and to continue with another variable corridor as in Figure 10.

We combine the variable corridors into a **clause gadget**, see Figure 11: the variable



(a) (b)

Figure 10 Negation gadget in blue, with two black variable loops. The incoming loop on the left, has a different truth assignment than the outgoing loop on the right, two cases shown in (a)/(b).

loops are shown in black, and the clause gadget in gray. There are three possible states of the clause gadget, depicted in red, turquoise, and violet (see Figure 11(b)). Each state forces exactly one of the variables' truth assignments to fulfill the clause assignment (all cases are shown in Figure 11(c)-(e)), giving a solution to the instance F.

A solution to an $m \times n$ Spiral Galaxies board can be verified in polynomial time. ◀

4 Non-Crossing Matching in Squared Grid Graphs

We define a *squared grid graph* as a modified grid graph where edges connect vertices at distance 2. From the proof of Theorem 3.1—interpreting the disks as graph vertices—we obtain a corollary:

▶ **Corollary 3.** *Non-crossing matching in squared grid graphs is NP-complete.*

5 Minimizing Centers in Spiral Galaxies for a Given Shape

In this section, we are given a (black) shape $\mathcal{S}$ on a Spiral Galaxies board, and we aim to find the minimum number of centers, such that there exist spiral galaxies with these centers that exactly covers the given shape $\mathcal{S}$. We show that this problem is NP-complete by a reduction—one more time—from PLANAR 1-IN-3-SAT.

▶ **Theorem 4.** *Minimizing the number of centers on a Spiral Galaxies board, such that Spiral Galaxies with these centers exactly cover a given shape $\mathcal{S}$ is NP-complete.*

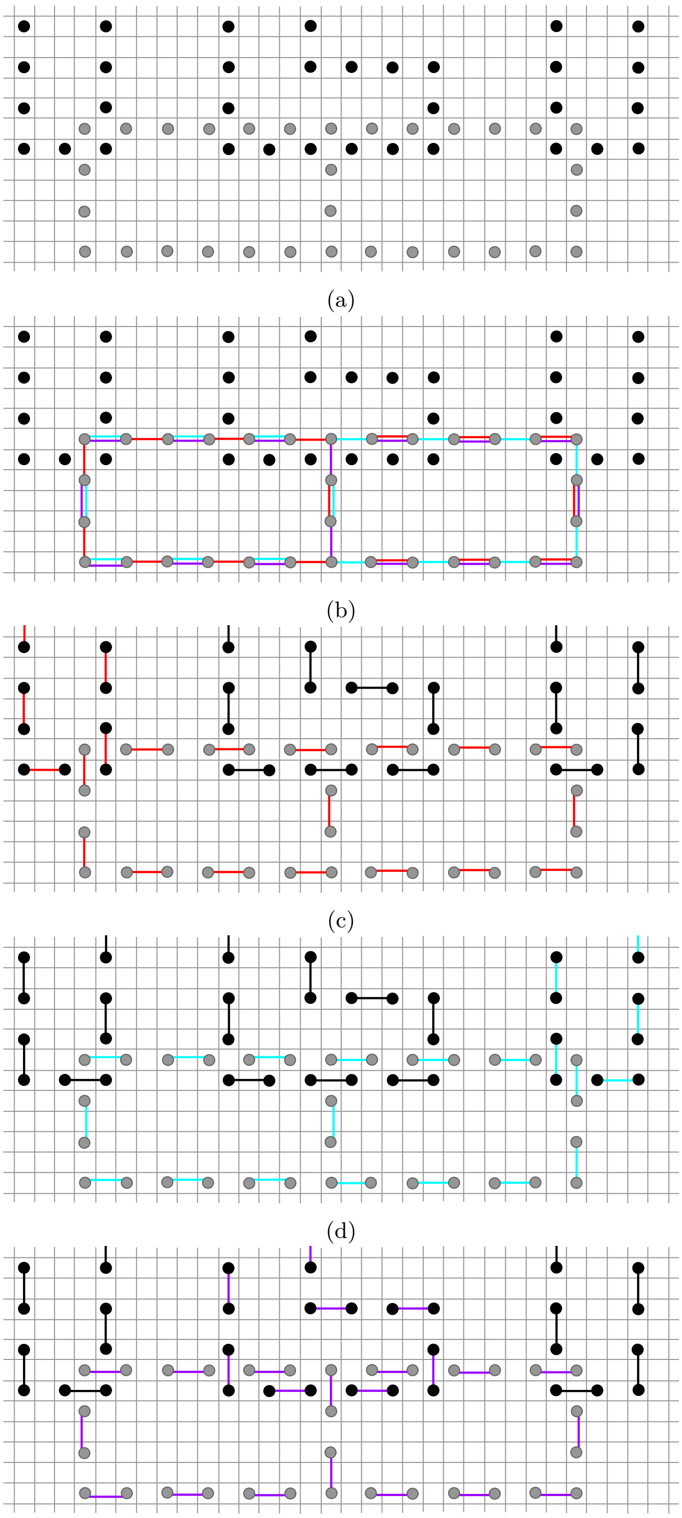
Proof. First, we introduce *local center gadgets*: thin constructions with unique shapes, which ensure that there must be at least one center in each of them; see Figure 12. The idea will now be to construct a shape which can be covered with exactly this set of centers if and only if the 1-in-3-sat instance is satisfiable.

To this end, we create *block gadgets*: 5×5 rooms which are connected to local center gadgets on two or three sides. We distinquish *straight blocks*, *corner blocks*, and *clause blocks*; see Figure 13. Next, we define the *fix gadget*: an alternating sequence of four local center gadgets and three block gadgets making a U-turn as in Figure 14.

We claim that in any chain of blocks that contains a fix gadget, each block must completely belong to a single galaxy. Indeed, suppose we have a fix gadget consisting of blocks A, B, and C, where A and C are corner blocks but B is a straight block, and suppose A is split among multiple galaxies, say a red one at the top and a blue one at the right (refer to Figure 14 (b)). Then the bottom center pixel of B must also be red, and hence the top center pixel of C must be red by summetry. But then C must be completely red; contradiction.

A **variable chain** consists of alternating local center gadgets and block gadgets starting and ending at *end gadgets*; see Figure 16 for a variable chain and Figure 15 for an end gadget. Each variable chain must include a fix gadget.

A clause is simply a single clause block where three variable chains meet: it can be solved without using an additional center if and only if precisely one of the three chains needs to use the block. The only missing ingredient now is a *split gadget*, which ensures we can make multiple variable chains with the same state. We let several variable chains end in a common area, which, similar to the end gadget, can be solved with one center only if all blocks are present, or if all blocks are absent; see Figure 17. Note that the distance between adjacent block gadgets may be adjusted as needed; this also allows us to negate variables. ◀



(a)

(b)

(c)

(d)

(e)

Figure 11 (a) Clause gadget, in gray, with the three incoming variable loops, in black. (b) shows the three possible states of the clause gadget, and (c)-(e) give each one assignment in the clause gadget, with the corresponding assignments of the variable loops, the (only) variable with a truth assignments that fulfills the clause is shown in the same color as the clause edges.

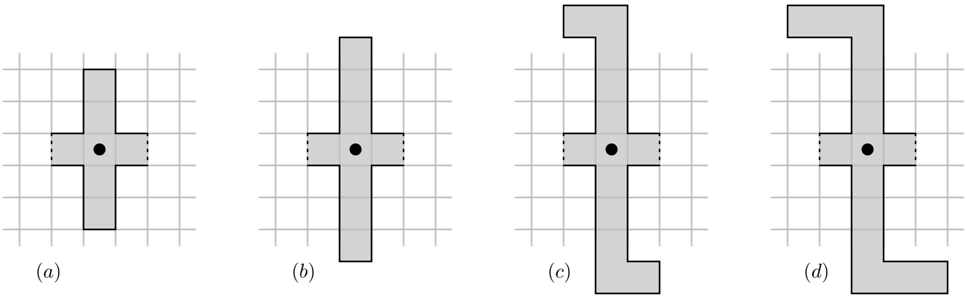


Figure 12 The local center gadget must have at least one center. (a-d) Different shapes ensure we cannot include them in larger galaxies.

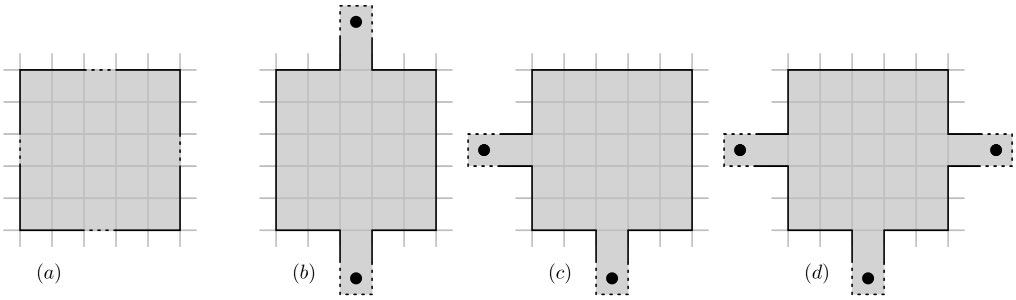


Figure 13 (a) The block gadget. (b) A straight block. (c) A corner block. (d) A clause block.

6 Some Spiral Galaxies Puzzles

The Spiral Galaxies board from Figure 18 solves for the black letters A, B, H, R, S, Z (and for disconnected galaxies also for the letter E). See Figures 19 and 20 for solutions.

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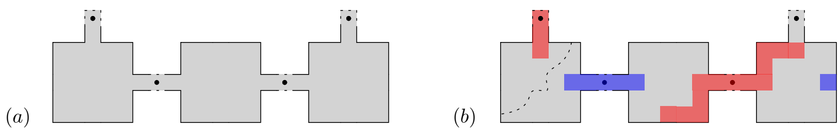

Figure 14 (a) The fix gadget. (b) If the left block is split among multiple galaxies: contradiction.

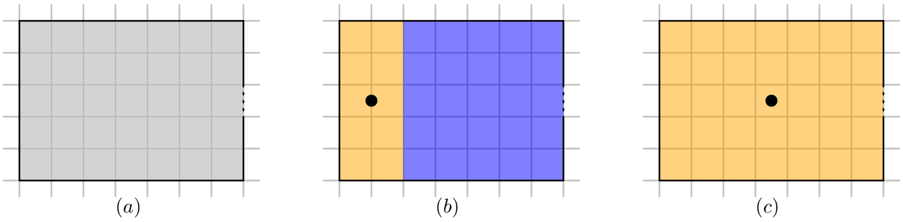

Figure 15 An end gadget has one center, whether or not a block is used by the galaxy next to it.

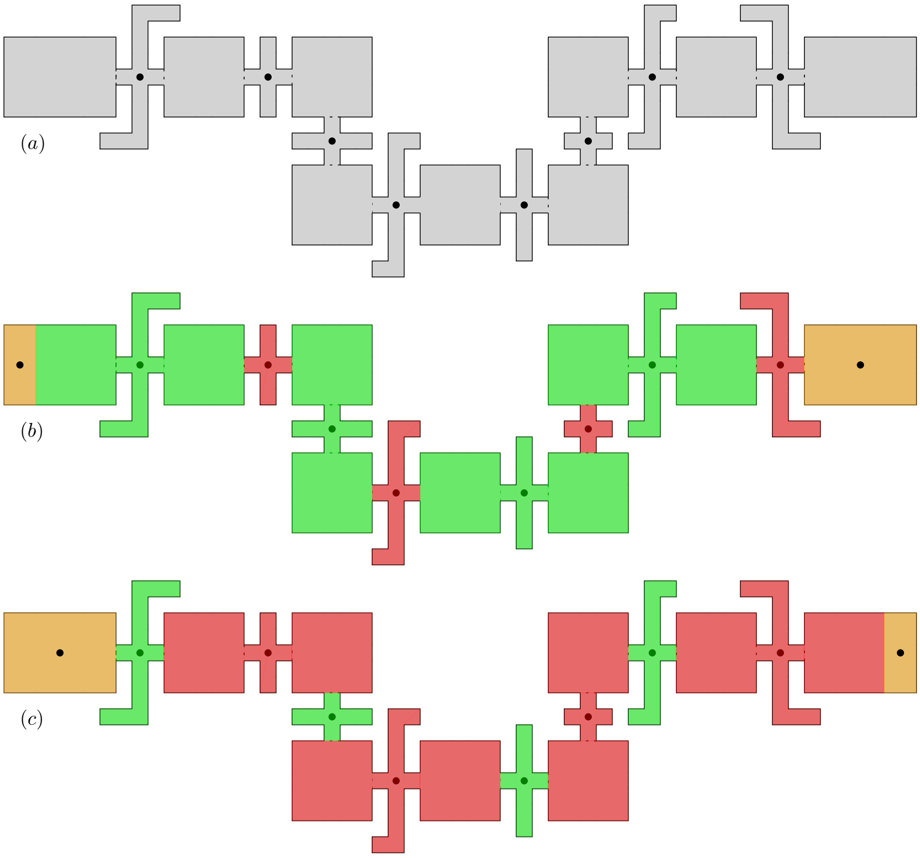

Figure 16 A variable chain and its two possible truth assignments.

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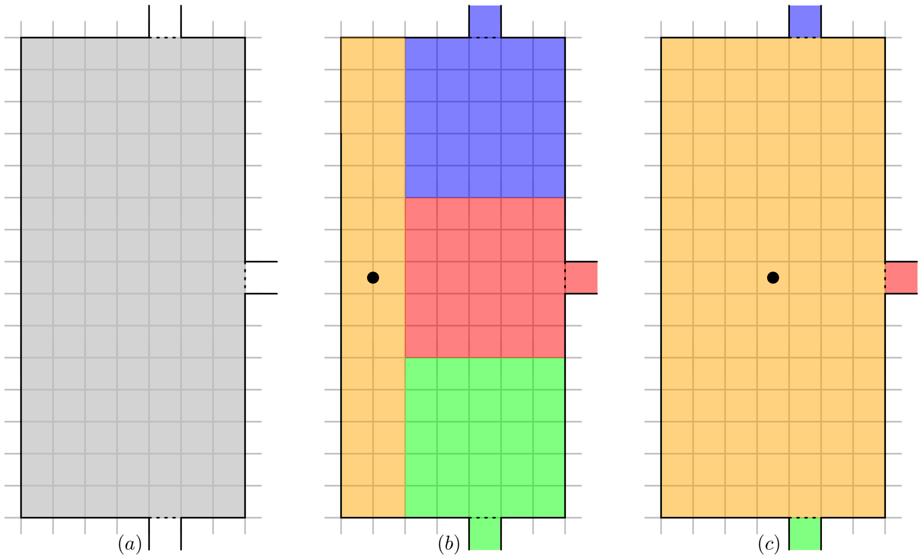


Figure 17 The split gadget is essentially three end gadgets, but can only be solved with a single center if either all three blocks are used or all three are not used.

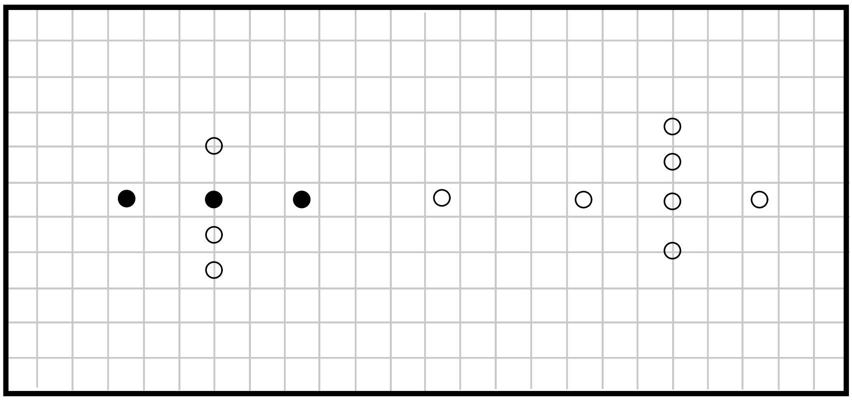


Figure 18 Puzzle that can be solved for letters A, B, H, P, R, S, Z (E for disconnected galaxies).

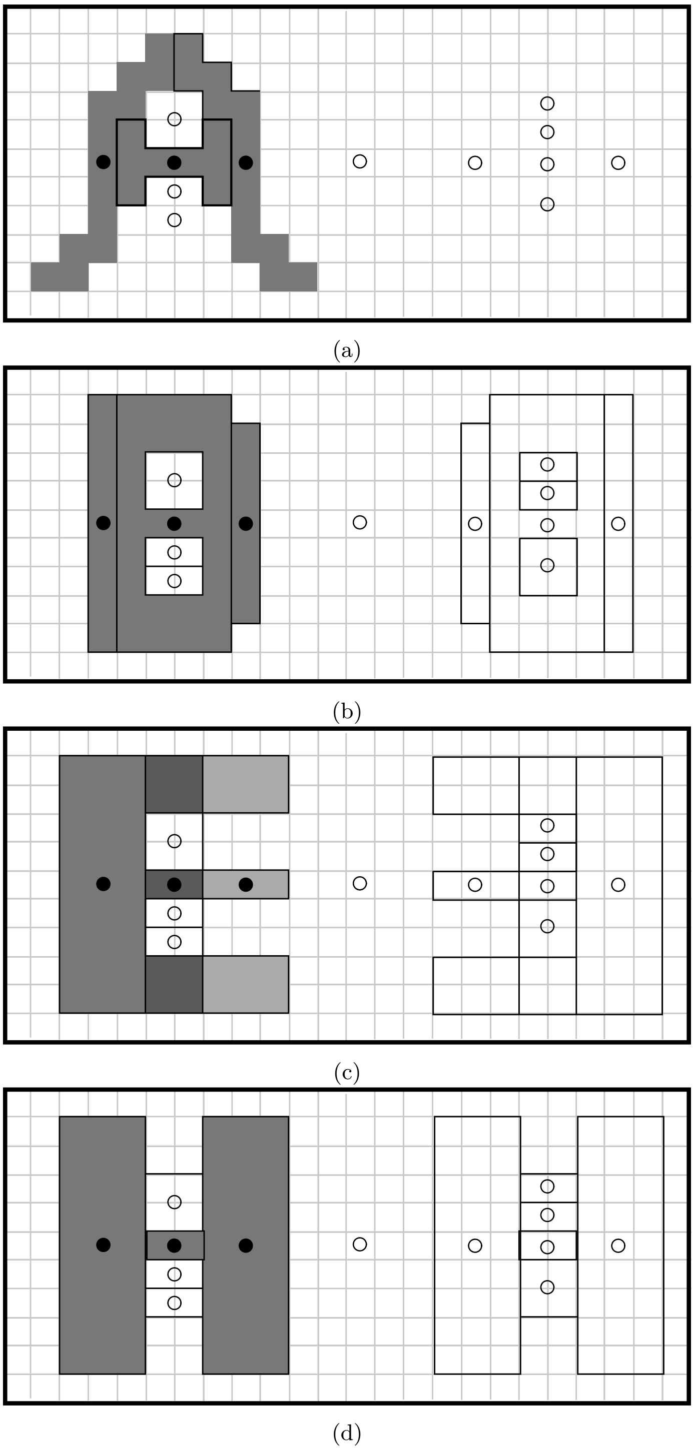
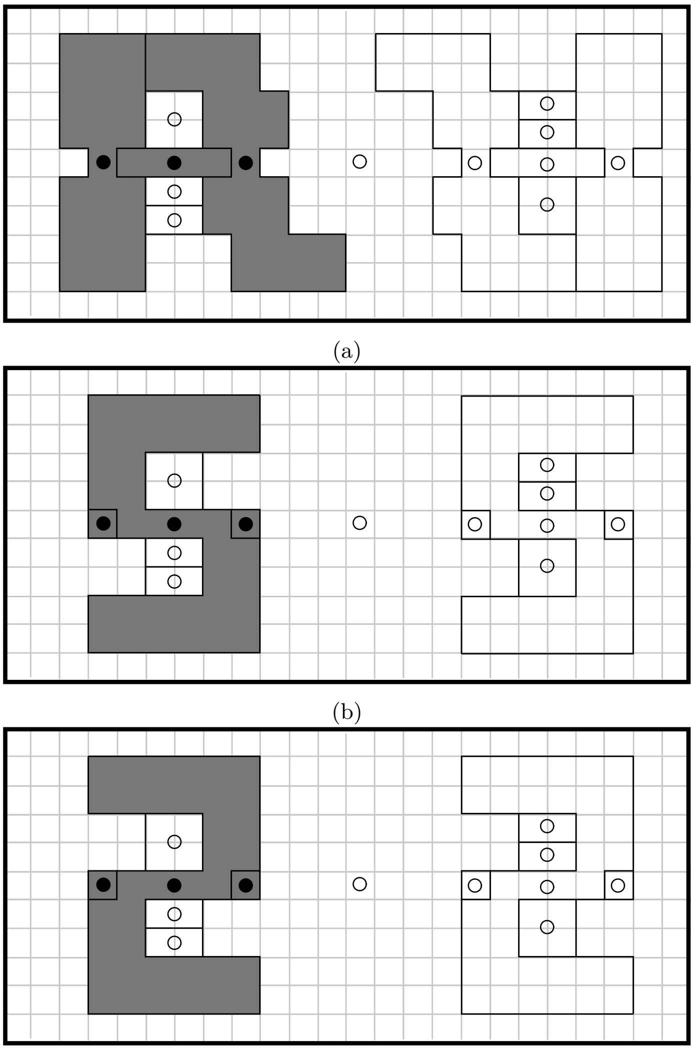


Figure 19 Puzzle solutions for letters A, B, E, H.



(a)

(b)

(c)

Figure 20 Puzzle solutions for letters R, S, Z.