La Salle's Invariant Set Theory

Asymptotic stability of a control system is often an important property to be determined.

Lyapunov's stability theorems studied above are often difficult to apply to establish this property, as it often happens that $\dot{V}$ (the derivative of the Lyapunov function candidate) is only negative semi-definite.

In this kind of situation, it is still possible to draw conclusions on asymptotic stability, with the help of the *invariant set* theorems, which are attributed to La Salle.

Definition 14 (Invariant set)**.** A set S is an invariant set for a dynamic system $\dot{x} = f(x)$ if every trajectory $x(t)$ which starts from a point in S remains in S for all time.

For example, any equilibrium point is an invariant set.

The domain of attraction of an equilibrium point is also an invariant set.

The invariant set theorems reflect the intuition that the decrease of a Lyapunov function V has to gradually vanish (In other words $\dot{V}$ has to converge to zero) because V is lower bounded.

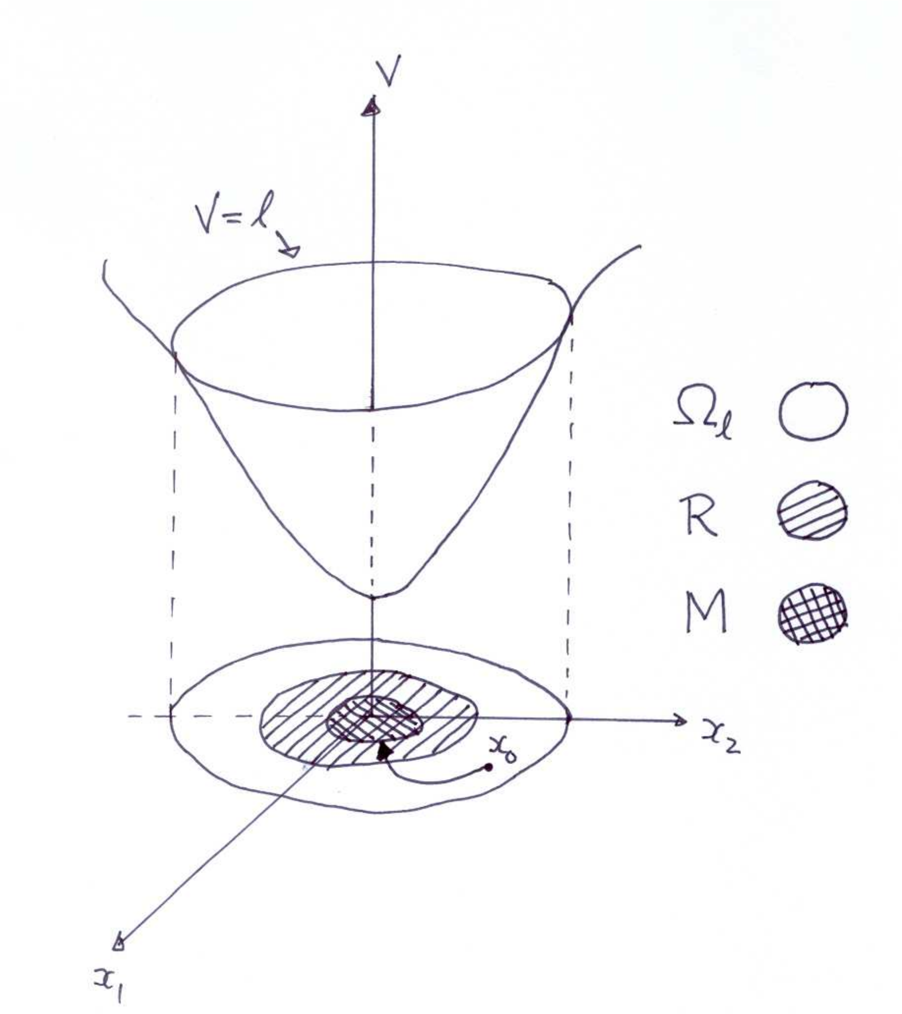
Theorem 5 (Local Invariant set theorem). *Consider an autonomous system of the form $\dot{x} = f(x)$, with f continuous and let $V(x) : \Re^n \to \Re$ be a scalar function with continuous first partial derivatives. Assume that*

- *for some $l > 0$, the set Ω_l defined by $V(x) \leq l$ is bounded.*
- *$\dot{V}(x) \leq 0$ for all x in Ω_l.*

Let R be the set of all points within Ω_l where $\dot{V}(x) = 0$ and M be the largest invariant set in R. Then, every solution $x(t)$ originating in Ω_l tends to M as $t \to \infty$.

Proof. See Slotine and Li (1991). □

In Theorem 5, the word largest means that M is the union of all invariant sets within R. The geometrical interpretation of the theorem is illustrated in Figure 1.



Convergence to the largest invariant set M

Notice that R is not necessarily connected, nor is the set M. Now we are ready to state the following important theorem.

Theorem 6 (La Salle's principle to establish asymptotic stability)**.** *Let $V(x) : \Re^n \to \Re$ be such that on $\Omega_l = \{x \in \Re^n : V(x) \leq l\}$ we have $\dot{V}(x) \leq 0$. Define $R = \{x \in \Re^n : \dot{V}(x) = 0\}$. Then, if R contains no other trajectories other than $x = 0$, then the origin 0 is asymptotically stable.*

Proof. It follows directly from Theorem 5. □

If Ω_l in Theorem 6 extends to the whole space $\Re^n$, then global asymptotic stability can be established.

La Salle's theory is a useful extension to Lyapunov theory. In summary,

- If $V(x)$ is a negative semi-definite in a region Ω_l where $V(x) \leq l$, then a solution starting in the interior of Ω_l remains there.

- If, in addition, no solutions (except the equilibrium $x = 0$) remain in R (the subset of Ω_l where $V(x) = 0$), then all solutions starting in the interior of Ω_l will converge to the equilibrium.

Example 10. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1(x_1^2 + x_2^2 - 4) - 4x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 4) \end{aligned} \tag{36}$$

and consider the equilibrium point $x = 0$. Given the function:

$$V(x) = x_1^2 + x_2^2 \tag{37}$$

its derivative $\dot{V}$ along any system trajectory is:

$$\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 4) \tag{38}$$

Notice that $\dot{V}(x) < 0$ within a circle of radius 2. Hence, using Lyapunov's stability theorem (Theorem 1) we infer that the origin is locally asymptotically stable. For $l = 2$, the region Ω_2 defined by $V(x) = x_1^2 + x_2^2 < 4$ is bounded. The set R is simply the origin 0, which is an invariant set. All the conditions of Theorem 6 are satisfied, hence any trajectory starting within the circle of radius 2 converge to the origin and this region is called the *domain of attraction*.

To run from Matlab >> ex10.m

Example 11. Consider the system:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^7(x_1^4 + 2x_2^2 - 10)\\ \dot{x}_2 &= -x_1^3 - 3x_2^5(x_1^4 + 2x_2^2 - 10)\end{aligned} \tag{39}$$

Notice that the set defined by $x_1^4 + 2x_2^2 = 10$ is invariant, since:

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10) = -4(x^10 + 12x^6)(x_1^4 + 2x_2^2 - 10) \tag{40}$$

which is zero on the set. The motion on this set is described by either of these equations:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^3\end{aligned} \tag{41}$$

We can infer that the invariant set represents a *limit cycle*, along which the state vector moves clockwise.

To check if the limit cycle is attractive, define the Lyapunov function candidate:

$$V(x) = (x_1^4 + 2x_2^2 - 10)^2 \qquad (42)$$

- This function represents the distance to the limit cycle.
- For an arbitrary positive number l, the region Ω_l, which surrounds the limit cycle, is bounded.
- The time derivative of V along trajectories of the system is given by:

$$\dot{V}(x) = -8(x_1^1 0 + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2 \qquad (43)$$

- Then $V(x)$ is strictly negative except if $x_1^4 + 2x_2^2 = 10$ or $x_1^{10} + 3x_2^6 = 0$, in which case $V(x) = 0$ (so that $V(x)$ is in fact negative semi-definite) .

- The first equation $x_1^4 + 2x_2^2 = 10$ simply defines the limit cycle.

- The second equation $x_1^{10} + 3x_2^6 = 0$ is satisfied only at the origin.

- Since both the limit cycle and the origin are invariant sets, the set M consists of their union.

- Then, all system trajectories starting in Ω_l converge either to the limit cycle, or to the origin.

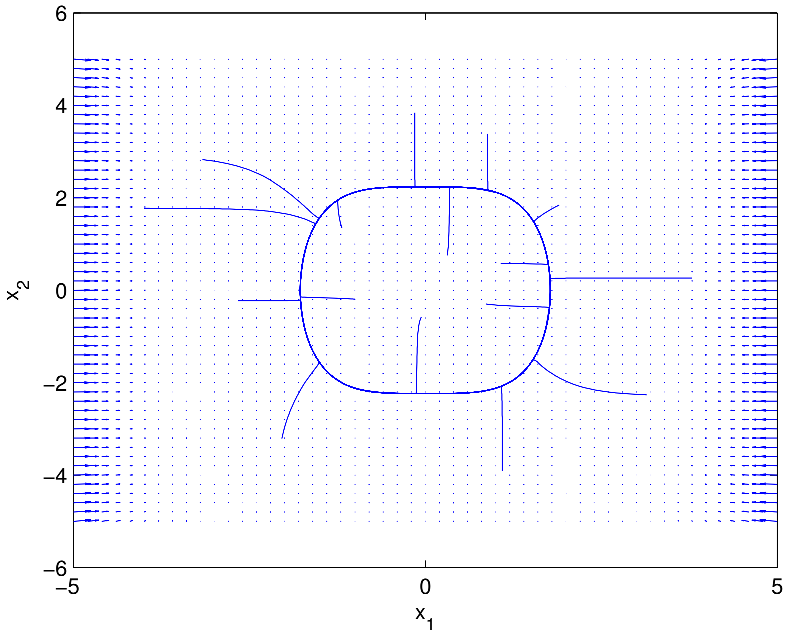
Consider for example, the case when $l = 100 - \epsilon$, for some small $\epsilon > 0$.

Notice that $V(0) = 100 > 100 - \epsilon$, such that the origin does not belong to Ω_l.

Then, while the expression for $\dot{V}$ is the same as the already given, now the set M is just the limit cycle.

Hence, since ϵ is arbitrary, applying once again the invariant set theorem shows that any trajectory starting from the region within the limit cycle, excluding the origin, actually converges to the limit cycle.

This implies that the origin is unstable!.



Convergence to an attractive limit cycle with an unstable equilibrium point at the origin

To run from Matlab >> ex11

Example 12. Consider the autonomous pendulum with friction:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned} \tag{44}$$

where $a, b > 0$ and the candidate Lyapunov function (the energy function in this case):

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2} x_2^2 \tag{45}$$

Let $D = \{x \in \Re^2 | -\pi < x_1 < \pi\}$. Then $V(x)$ is positive definite in D. The derivative $\dot{V}(x)$:

$$\dot{V}(x) = a \dot{x}_1 \sin(x_1) + x_2 \dot{x}_2 = -b x_2^2 \tag{46}$$

is negative semidefinite in D. It is not negative definite since $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective the value of x_1.

Therefore, using Lyapunov's stability theory it is only possible to conclude that $x = 0$ is stable. However, we *know* that $x = 0$ is asymptotically stable.

- In this case, we can use La Salle's Theorem 2 to prove asymptotic stability.

- Note that $R = \{x \in D | x_2 = 0\}$, so that $\dot{V}(x)$ vanishes at $x \in R$.

- Let $x(t)$ be a solution that belongs to R, so that $x_2(t) = 0$. This implies by looking at the equation for $\dot{x}_2$ that $\sin(x_1(t)) = 0$, and hence that $x_1(t) = 0$.

- So that the only solution that can stay identically in R is the trivial solution $x(t) = 0$.

- Using Theorem 6 we can conclude that $x = 0$ is asymptotically stable.