Chapter 1

Introduction

In these notes we will try to give an introduction to the theory of group representations and we will demonstrate the usefulness of this theory by investigating several problems from image science. This group theoretical approach was motivated by the observation that many problems in image processing are very regular. In this introduction we will, as an example, study the well-known edge-detection problem for 2-D images.

1.1 Edge Detection

In a first attempt we can formulate this problem as follows:

A point in an image is an *edge point* if there is a line through this point such that all points on one side of this line are dark and all points on the other side are light. The problem to find all edge points in a given image is called the *edge detection problem.*

This definition of an edge point contains one degree of freedom since we did not specify the direction of the line that separates the black and the white points. There is thus in infinite number of different types of edge points: one type of edge point for each direction of the edge. We reformulate the edge detection problem as follows:

Define L_ψ as the line that goes through the origin of the coordinate system and has an angle ψ with the x-axis. By p_ψ we denote the function that has the value one on the left side of L_ψ and the value zero on the other side. The set of step edges, denoted by $\mathcal{E}$, is the set of all functions p_ψ with $0 \leq \psi < 2\pi$.

Given a point P in the image with gray value distribution f in a neighborhood we want to decide if $f \in \mathcal{E}$ or $f \notin \mathcal{E}$. In the first case we say that P is an edge point.

This formulation of the problem reveals that the set of edge patterns $\mathcal{E}$ is not just an arbitrary collection of patterns. On the contrary, it is highly structured, or symmetric, since we can generate all elements in this set by rotating a fixed pattern around the origin. The fact that the set of patterns that we want to detect is very regular is so important that we introduce a special concept to describe this type of regularity.

1.2 Invariant Pattern Classes

In the general case we consider a set of functions $\mathcal{C}$. The elements in $\mathcal{C}$ are the patterns we want to detect in the image. The fundamental pattern recognition problem is thus to decide if a given pattern p is an element of $\mathcal{C}$ or not.

Without any restriction on $\mathcal{C}$ we can solve this problem only by comparing p with all elements in the pattern class $\mathcal{C}$. However, the situation changes drastically if $\mathcal{C}$ is structured. The interesting structures of $\mathcal{C}$ are based on two properties:

Domain: First there is a common domain for all functions in $\mathcal{C}$, i.e. a common set on which all the pattern in $\mathcal{C}$ are defined. In the edge-detection problem this domain is the unit disk.

Transformation: Furthermore the patterns in $\mathcal{C}$ are all coupled by a transformation rule. This means that we can generate all patterns in the class by applying the transformation rule to a fixed pattern in the class. In the case of the step edges the transformation rule is defined in terms of rotations.

The essential part in this construction is the transformation rule. This rule links all the patterns in the pattern class together. We will sometimes also say that all the patterns in the class are essentially the same.

One of the main goals of these lectures is the exact definition and the investigation of this type of pattern class.

1.3 Pattern Detection

Now we return to the edge detection problem and we demonstrate that there is a connection between the description of the pattern class and the detection of these patterns in an image.

We saw that the step edges were functions of the form $p_\psi(r,\varphi)$, where (r,φ) are the polar coordinates of a point. We ignore the different character of the variables r,φ and ψ and write $p(r,\varphi,\psi)$. If we fix the spatial variables r and φ then we can consider p as a function of the transformation variable ψ alone.

A Fourier decomposition of p leads to the following *weighted group averages*

$$P_k(r,\varphi) = \int_0^{2\pi} p(r,\varphi,\psi)\mathrm{e}^{ik\psi}\, d\psi \tag{1.1}$$

Note that this defines a new set of (complex-valued) patterns defined on the same domain. These patterns define the pattern class completely since we can recover the original patterns by forming linear combinations of the new patterns P_k:

$$p(r,\varphi,\psi) = \sum a_k \mathrm{e}^{ik\psi} P_k(r,\varphi) \tag{1.2}$$

This decomposition of the patterns has the advantage that the spatial variables r and φ and the transformation variable ψ are separated. Furthermore we see that the different components $\mathrm{e}^{ik\psi}P_k(r,\varphi)$ in the decomposition transform nicely when we go from $p(r,\varphi,\psi_0)$ to $p(r,\varphi,\psi_0+\psi_1)$: every component is simply multiplied by a complex factor.

Now assume that we have an unknown pattern p_u and we want to decide if p_u is a member of the pattern class. To solve this problem we compute again P_k and find:

$$\begin{aligned} P_k(r,\varphi) &= \int_0^{2\pi} p(r,\varphi,\psi)\mathrm{e}^{ik\psi}\, d\psi && (1.3)\\ &= \int_0^{2\pi} p(r,\varphi+\psi,0)\mathrm{e}^{ik\psi}\, d\psi && (1.4) \end{aligned}$$

$$= \quad e^{-ik\varphi} \int_0^{2\pi} p(r,\psi,0) e^{ik\psi} \, d\psi \tag{1.5}$$

The weighted average over the pattern class is thus (up to a constant complex factor of magnitude one) equal to the *weighted spatial average* over one pattern in the class.

We can use this relation to decide if a pattern p_u is equal to one pattern $p(r,\varphi,\psi)$ in the pattern class by doing the following computations:

- First we compute the weighted spatial averages from the prototype pattern. These averages are, up to a complex factor, equal to the weighted class averages over the pattern class. The result are complex functions $P_k(r,0)$.
- Then we compute the spatial weighted averages over the unknown pattern. As result we get complex functions $U_k(r)$.
- If now $P_k(r,0) = e^{ik\alpha}U_k(r)$ for a constant α and all k then we know that the unknown pattern was a pattern in the given pattern class.

We found thus that the weighted class averages and the weighted spatial averages were very useful in the description and the analysis of the pattern class. The usefulness of this approach depended of course on the clever selection of the weight function $e^{ik\psi}$ and one of the main purposes of these notes is the construction of similar useful weight functions for a number of different types of invariant pattern classes.