Chapter 3

Representations of Groups

In this chapter we will introduce the concept of a representation of a group. The methods introduced here are of fundamental importance in the study of symmetries and they have been applied in such different fields as quantum mechanics and the theory of special functions. Apart from these "applications" the theory is also useful for the study of abstract groups since these abstract groups are linked via representations to well-known matrix groups.

In the next section we will first introduce the algebraic concepts involved and then we will specialize the concepts to the case where the groups also have a topological structure.

3.1 Algebraic Representations

Definition 3.1 Let G be any group, X a complex vector space and denote by $GL(X)$ the group of invertible, linear mappings that carry X into itself.

1. Then we define a *representation* as a homomorphism $T : G \to GL(X); g \mapsto T(g)$.
2. X is called the *representation space.*
3. $T(g)$ is called a *representation operator.*

We call a representation in the sense of definition 3.1 an *algebraic* representation in contrast to the *continuous* representations to be introduced in the next section 3.12 (see also [37] page 151). In the definition of continuous representations we will mainly require that the operators T are continuous, a property which is of course coupled to the topological properties of the group and the vector space.

The vector space is finite-dimensional in almost all applications of interest to us. In this case we define:

Definition 3.2

1. If the representation space X is a finite-dimensional vector-space then we call the representation *finite* or *finite-dimensional.* Otherwise it is called *infinite* or *infinite-dimensional.*
2. The dimension of X is called the *dimension of the representation.*

3. Assume T is an n-dimensional representation and $e_1, \ldots, e_n$ is an arbitrary but fixed basis of X. Relative to this basis $T(g)$ can be described by a matrix

$$t(g) = \begin{pmatrix} t_{11}(g) & \cdots & t_{1n}(g) \\ \vdots & & \vdots \\ t_{n1}(g) & & t_{nn}(g) \end{pmatrix}$$

$t(g)$ is called the *matrix of the representation* and the functions $t_{ij}(g)$ are called its *matrix elements.*

If we do not say otherwise, then the representations are assumed to be finite-dimensional. From the definition we find the following properties of a representation:

Theorem 3.1 1. $T(e) = id$, where e is the identity in G and id the identity mapping in $L(X)$.

2. $T(g^{-1}) = T(g)^{-1}$

The first property is a general property of a homomorphism. To see the second equation compute: $id = T(e) = T(gg^{-1}) = T(g)T(g^{-1})$.

For the corresponding matrices we get:

Theorem 3.2 1. If t is a matrix representation of G then we have the matrix equation:

$$t(g_1 g_2) = t(g_1)t(g_2).$$

2. For the matrix elements this becomes:

$$t_{ij}(g_1 g_2) = \sum_k t_{ik}(g_1)t_{kj}(g_2)$$

3. The matrix $t(e)$ is the identity matrix.

By an easy calculation it can be shown that the following functions are examples of one-dimensional representations:

Examples 3.1 1. Let $G = \mathbf{R}$ be the additive real group and $k \in \mathbf{C}$. The function: $T_k(x) = e^{kx}$ is a one-dimensional representation. We write e^{kx} instead of $\left(e^{kx}\right)$ but we have to remember that $T_k(x)$ is a function $T_k(x) : \mathbf{C} \to \mathbf{C}$ with $T_k(x)(\zeta) = e^{kx}\zeta$.

2. Let $G = \mathbf{C}$ be the additive complex group and let $k_1, k_2 \in \mathbf{C}$ be two complex constants. Then $z = x + iy \mapsto e^{k_1 x}e^{k_2 y}$ is a one-dimensional representation. As in the previous case we have: $T_{k_1 k_2}(x + iy)\zeta = e^{k_1 x}e^{k_2 y}\zeta$.

3. Let $G = \mathbf{R}_0$ be the non-zero multiplicative real group, $k \in \mathbf{C}$ be a complex constant and $\epsilon \in \{0, 1\}$. Then $x \mapsto e^{k(\ln|x|)}(\text{sign } x)^{\epsilon}$ is a one-dimensional representation of $\mathbf{R}_0$.

4. Let $G = SO(2)$ be the circle group, ϕ be the rotation angle and $n \in \mathbf{Z}$. Then $\phi \mapsto e^{in\phi}$ is a one-dimensional representation of $SO(2)$.

A very important property of representations is connected to subspaces of X which are left invariant under all transformations $T(g)$:

Definition 3.3 1. A subspace $Y \subset X$ is called an *invariant subspace of* T if Y is left invariant under all transformations $T(g), g \in G$, i.e.

$$T(g)y \in Y \text{ for all } g \in G \text{ and all } y \in Y$$

2. A representation is called *irreducible* if $\emptyset$ and X are the only invariant subspaces of X.
3. A representation is *reducible* if it is not irreducible.

By induction it is easy to prove that if T is a representation on a finite-dimensional space X then there is a subspace $Y \subset X$ on which T is irreducible.

There are of course infinitely many representations for each group but many of them are essentially equal to each other. In the next definition we specify what we mean by essentially equal representations:

Definition 3.4 Assume S and T are two representations of a group G in the spaces X and Y. S and T are called *equivalent* ($T \sim S$) if there is an isomorphism $A : X \rightarrow Y$ such that $AT(g) = S(g)A$ for all $g \in G$.

The operator A in the previous definition is only one example of a function that connects two representations. In the general case we define:

Definition 3.5 Let S and T be representations in the spaces X and Y respectively. Assume further that $A : X \rightarrow Y$ is a linear operator. We say that A is an *intertwining operator for* T *and* S if $AT(g) = S(g)A$ for all $g \in G$.

An intertwining operator is obviously a generalization of the equivalence concept since S and T are equivalent representations in the case where A is an isomorphism and where A is an intertwining operator for S and T.

For finite-dimensional representations (and especially one-dimensional representations) we get the following characterization of equivalent representations:

Theorem 3.3 1. Two finite-dimensional representations are equivalent if there are two bases in X and Y in which their matrices coincide (This means especially that X and Y have the same dimension).

2. Two finite-dimensional matrix representations t and s are equivalent if there is a matrix A such that $At(g)A^{-1} = s(g)$ for all $g \in G$.
3. Equivalent one-dimensional representations are defined by the same function $t(g)$.

Irreducible representations can be characterized by intertwining operators. This result is known as Schur's Lemma:

Theorem 3.4 1. Assume S and T are irreducible representations of the group G in the spaces X and Y. Assume further that $A : X \rightarrow Y$ is an intertwining operator for T and S (i.e. $AT(g) = S(g)A$ for all $g \in G$). Then A is either an isomorphism or $A = 0$.

2. Assume T is a finite-dimensional, irreducible representation of G in X. Then every linear operator $B : X \to X$ that satisfies: $BT(g) = T(g)B$ for all $g \in G$ has the form $B = \lambda id$ where λ is a complex constant.
3. An irreducible, finite-dimensional representation of a commutative group is one-dimensional.

Proof: The image $L = AX$ of X under A is a subspace of Y. L is invariant under $S(g)$ because $S(g)Ax = A(T(g))x = Ax' \in L$. S is irreducible and L is therefore the null-space or the whole space Y. In the first case $A = 0$, in the second case we find that A is surjective and we show that A is also injective. We note that the kernel is an invariant subspace and therefore it is either equal to the null-space or to the whole space. The second case is impossible since A would otherwise be the null-space. This proves the first part of the theorem.

To show the second part we note that B is a linear operator and therefore it has at least one eigenvalue λ. We define $A = B - \lambda id$. A is intertwining but it is not an isomorphism and therefore A is the null-operator by the previous part of the theorem.

If T is an irreducible, finite-dimensional representation of the commutative group G then $T(g_0)$ satisfies

$$T(g)T(g_0) = T(gg_0) = T(g_0g) = T(g_0)T(g)$$

for all $g \in G$. Therefore we find that $T(g_0) = \lambda_0 id$ by the previous part. Every subspace of X is therefore invariant under all $T(g)$ and the dimension of the space must therefore be one.

We will now describe how we can decompose a given representation into a number of simpler components. For this purpose we introduce the direct sum of representations:

Definition 3.6 Assume G is a group and $X_1, \dots, X_n$ are vector spaces. Assume further that T^i are representations of G in X_i. Then we define the *direct sum* T of the representations T^i as follows: As representation space we take the direct sum of the vector spaces and as mapping we define $T(g)(x) = T(g)(x_1 + \ldots + x_n) = T^1(g_1)x_1 + \ldots + T^n(g_n)x_n$.

Now assume that all the representations are finite-dimensional. Then we select a basis in each vector space and get a basis in the sum by the union of all the basis elements. If we group them together in a correct order then we find that the matrix representation of the sum is given by

$$t(g) = \begin{pmatrix} t^1(g) & & & & & & \\ & t^2(g) & & & & & \\ & & t^3(g) & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & t^n(g) \end{pmatrix}$$

We introduced the direct sum of representations as a tool to build a new representation from a number of given representations. In the following definition we invert this process by introducing a special class of representations that can be decomposed into a number of simpler components.

Definition 3.7 A representation is called *completely reducible* if it is the direct sum of a finite number of irreducible representations .

The matrices of a finite-dimensional, completely reducible representation can thus be simultaneously diagonalized so that the diagonal matrices define irreducible matrix representations.

A reducible representation is not necessarily completely reducible as the following example shows:

Examples 3.2 Take as group the real numbers **R** and as representation the mapping $x \mapsto T(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, i.e. each number is mapped into a translation. Then we find for an element $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ of an one-dimensional invariant subspace M the equations:

$$T(x)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 + x\xi_2 \\ \xi_2 \end{pmatrix} = \lambda(x)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \lambda(x)\xi_1 \\ \lambda(x)\xi_2 \end{pmatrix}$$

and from this we find that the only invariant subspace M is the space given by the vectors $\begin{pmatrix} \xi_1 \\ 0 \end{pmatrix}$. The representation is thus reducible but not completely reducible.

Up to now we used only the basic properties of groups and vector spaces. In the next definition we will now assume that the representation space X is also equipped with a scalar product $\langle\ ,\ \rangle$ and we will study representations which are compatible with the scalar product, i.e. representations that leave the geometry in X invariant.

Definition 3.8 Assume T is a representation of the group G in X. A representation $T^\star$ is called the *adjoint representation* if we have for all $x, y \in X$ and all $g \in G$

$$\langle T(g)x, T^\star(g)y\rangle = \langle x, y\rangle\,.$$

For the matrices of adjoint representations we find the following characterization:

Theorem 3.5 Assume T is a finite-dimensional representation with matrix representation $t(g) = (t_{ij}(g))$. Then the adjoint representation $S(g)$ has the matrix representation $s(g) = (s_{ij}(g))$ with $t_{ij}(g^{-1}) = \overline{s_{ji}(g)}$.

Of special importance are the representations that leave the metrical properties of the vector spaces invariant. These representations are the selfadjoint, or unitary representations:

Definition 3.9

1. Assume X is a pre-Hilbert space with a scalar product $\langle\ ,\ \rangle$. Then we say that a representation T is *unitary* if

$$\langle T(g)x, T(g)y\rangle = \langle x, y\rangle$$

for all $g \in G$ and all $x, y \in X$.

2. An $n \times n$ matrix t is *unitary* if $t^\star t = E_n$ where $t^\star$ is the transposed, conjugate complex of t.

By a simple calculation we find:

Theorem 3.6 1. The matrices $t(g)$ of a unitary representation T in an orthonormal basis are unitary.

2. If $t(g)$ is the matrix of the unitary representation T then we have:

$$t^*(g) = t(g)^{-1} = t(g^{-1})$$

$$t_{ij}(g^{-1}) = \overline{t_{ji}(g)}$$

If a vector space is not a pre-Hilbert space or if the original scalar product is not compatible with a given representation then it is sometimes possible to introduce a new scalar product under which the representation is unitary. In this case we define:

Definition 3.10 A representation T in a space X is called *equivalent to a unitary representation* if there is a scalar product on X under which T becomes a unitary representation.

For these representations we have:

Theorem 3.7 If a finite-dimensional representation is equivalent to a unitary representation then it is completely reducible.

Proof: If M is an invariant subspace then $M^\perp$ is also invariant. Furthermore we know that any finite representation has an invariant subspace on which T is irreducible. Assume M_1 is such a subspace. If $M_1 = X$ we are through, otherwise we get $M_1^\perp \neq \{0\}$ and $M_1^\perp$ is also invariant under T. Now do the same construction on M_1 instead of M and get a subspace M_2 on which T is irreducible. If M_1 and M_2 span X then we are through, otherwise we do the same for the remaining space.

As a last construction in this section we introduce the character of a representation. It will be used later on (see section 5.2) as a tool to characterize representations. Here we derive only the basic properties of the character.

Definition 3.11 1. If $t = (t_{ij})$ is an $n \times n$–matrix then we define the *trace* of t as: $tr(t) = \sum_{i=1}^{n} t_{ii}$. The trace is thus the sum of all diagonal elements.

2. Assume that T is a finite-dimensional representation and t its matrix representation. We define the *character of the representation* as the function $\chi_T : G \to \mathbf{C}; g \mapsto \chi_T(g) = tr(t(g))$.

Note the matrix representation t of T depends on the selected basis of X but since a change of bases with a transformation matrix A results in the equivalent matrix representation $A^{-1}tA$ and since $tr(A^{-1}tA) = tr(t)$ we find the χ_T is independent of the basis chosen.

From the properties of the trace we find with some simple computations:

Theorem 3.8 For finite-dimensional representations we have the following properties:

1. If T and S are equivalent representations then $\chi_T = \chi_S$.
2. Characters are constant on conjugacy classes in G.
3. If T is unitary then $\chi_T(g^{-1}) = \overline{\chi_T(g)}$.

Proof: The first part was already mentioned in the definition of a character. For the second we note that g_1 and g_2 are conjugate if there is a $g \in G$ such that $g_2 = gg_1g^{-1}$ and we get: $\chi_T(g_2) = tr(t(g_2)) = tr(t(gg_1g^{-1})) = tr(t(g)t(g_1)t(g^{-1})) = tr(t(g)t(g_1)t(g)^{-1}) = tr(t(g_1)) = \chi_T(g_1)$.

If T is unitary then we have $t(g^{-1}) = t^*(g)$ and since we have $tr(t^*) = \overline{tr(t)}$ we see at once the last part of the theorem.

3.2 Continuous Representations

In the previous section we introduced representations of groups in vector spaces and pre-Hilbert spaces. In this section we will concentrate on the case where G is a topological group and where X is a topological vector space or even a Hilbert space.

Definition 3.12

1. A *topological vector-space* is a vector space which is also a Haussdorf topological space and which satisfies the following conditions:
   a) Addition is continuous, i.e. the mapping $(x, y) \mapsto x + y$ is a continuous function on $X \times X \to X$.
   b) Scalar multiplication is continuous, i.e. $(c, x) \mapsto cx$ is a continuous function on $\mathbf{C} \times X \to X$.
2. Assume G is a topological group and X is a topological vector space. A mapping $T : G \to GL(X)$ is a *continuous representation* if T is an algebraic representation and if $(x, g) \mapsto T(g)x$ is a continuous mapping of $X \times G \to X$.
3. Assume S and T are two continuous representations in two spaces X and Y then we call S and T *equivalent* if there is a homeomorphism $A : X \to Y$ which is linear and one-to-one and which satisfies:
$$AS(g) = T(g)A$$
for all $g \in G$.
4. A continuous representation T is called *irreducible* if $\{0\}$ and X are the only closed, invariant subspaces of T.

We will in the following only speak of representations when we mean continuous representations. The definition of representations and algebraic representations are identical up to the following, more or less technically motivated, differences:

1. The mapping T is continuous in g and x
2. The intertwining operator A in the equivalence definition must be a homeomorphism and
3. in the definition of irreducible representations we added the condition that the invariant subspaces must be closed.

From the definition we see direct:

Theorem 3.9 The matrix elements $t_{ij}(g)$ and the character $\chi_T(g)$ are complex-valued, continuous functions on G.

We can now show that representations of compact groups on Hilbert spaces have an especially simple structure, they are completely reducible as will be shown in the following two theorems:

Theorem 3.10 A representation T of a compact group in a Hilbert space H is equivalent to a unitary representation.

We note that there is a Haar integral $\int_G f(g)\, dg$ on G since G is compact. We have to show that H can be equipped with a scalar product under which T is unitary. It is easy to see that such a scalar product is given by

$$\langle x, y\rangle = \int_G \langle T(g)x, T(g)y\rangle_1 \; dg$$

where $\langle x, y\rangle_1$ is the original scalar product in H.

For unitary representations one can show that the orthogonal complement of an invariant subspace is also invariant and we get:

Theorem 3.11 All finite-dimensional representations of a compact group are completely reducible.

The next theorem, stated without proof, shows that it is sufficient to consider only finite-dimensional, irreducible, unitary representations of a compact group (see [37]):

Theorem 3.12 All irreducible, unitary representations of a compact group are finite-dimensional.

Up to now we derived a number of properties of representations and we listed some examples of algebraic representations. In the following theorem we will now construct **all** irreducible representations of the group of additive, real numbers $\mathbf{R}$. This group is commutative and we know therefore from Schur's lemma (see 3.4) that these representations are all one-dimensional. It is therefore sufficient to construct all one-dimensional representations of $\mathbf{R}$. This is done in the next theorem:

Theorem 3.13 All one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}$ are given by $f(x) = e^{kx}$ with a complex constant $k \in \mathbf{C}$.

Note again that e^{kx} stands for the 1×1 matrix $\left(e^{kx}\right)$.

Proof: Assume that f is a one-dimensional representation of $\mathbf{R}$. From the properties of representations we find the following equations for f:

$$f(0) = 1, \tag{3.1}$$

$$f(x + y) = f(x)f(y) \text{ for all } x, y \in \mathbf{R}. \tag{3.2}$$

and

$$f(-x) = f^{-1}(x) \text{ for all } x \in \mathbf{R} \tag{3.3}$$

From the last equation 3.3 we get:

$$f(x) \neq 0 \text{ for all } x \in \mathbf{R}. \tag{3.4}$$

Now take a point $x_0 \in \mathbf{R}$ and a function $g(x)$ with the following properties:

- g is infinitely differentiable,
- g is zero outside a neighborhood of x_0 and
- there is a constant $c \neq 0$ such that:

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = c \neq 0. \tag{3.5}$$

The existence of such a function follows from equation 3.4. From equation (3.5) we find

$$\int_{-\infty}^{\infty} f(x+y)g(y)\,dy = f(x)\int_{-\infty}^{\infty} f(y)g(y)\,dy = f(x)\cdot c$$

and

$$f(x) = \frac{1}{c}\int_{-\infty}^{\infty} f(x+y)g(y)\,dy = \frac{1}{c}\int_{-\infty}^{\infty} f(y)g(y-x)\,dy$$

From the properties of g it follows that the last integral is infinitely differentiable. From equation (3.2) we conclude that f is a solution of the differential equation:

$$f'(x) = kf(x).$$

We conclude that $f(x) = \mathrm{e}^{kx}$ for some complex constant k.

The previous theorem shows that the exponential function is characterized by the transformation property $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$. Another way to express the same fact is to say that the exponential function is characterized by its property of defining irreducible representations of $\mathbf{R}$. We saw also how irreducible representations define a link between groups and functions that behave nicely under the group operations. We found also in the case $G = \mathbf{R}$ that these functions are characterized by this transformation property. One application of the theory of group representations is thus the investigation of this type of problem in the general group theoretical setting, i.e. given a group G what are the special functions connected to G via group representations.

For other commutative groups we could derive similar results, all involving the exponential function in some way. In the following examples we give a list of irreducible representations of different commutative groups (see [37]):

Examples 3.3 1. All unitary, one-dimensional representations of the group $\mathbf{R}$ are given by $x \mapsto f(x) = \mathrm{e}^{itx}$ with a real constant $t \in \mathbf{R}$.

2. All one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}^n$ are of the form $f(x_1,\ldots,x_n) = \mathrm{e}^{k_1x_1+\ldots+k_nx_n}$ with complex constants $k_i \in \mathbf{C}$.
3. All unitary one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}^n$ are given by $f(x_1,\ldots,x_n) = \mathrm{e}^{i(t_1x_1+\ldots+t_nx_n)}$ with real constants $t_i \in \mathbf{R}$.
4. All one-dimensional representations of the group $\mathbf{C}^n$ are given by

$$z \mapsto f(z) = f(z_1,\ldots,z_n) = \mathrm{e}^{p_1z_1+q_1\overline{z_1}+\ldots+p_nz_n+q_n\overline{z_n}}$$

with complex constants $p_i, q_i \in \mathbf{C}$.

5. All unitary one-dimensional representations $z \mapsto f(z)$ of the group $\mathbf{C}^n$ are given by $f(z_1,\ldots,z_n) = \mathrm{e}^{p_1z_1-\overline{p_1z_1}+\ldots+p_nz_n-\overline{p_nz_n}}$ with complex constants $p_i \in \mathbf{C}$.

6. All one-dimensional representations of the 2-D rotation-group $SO(2)$ are given by $\alpha \mapsto f(\alpha)$ with $f(\alpha) = e^{im\alpha}$ where $m \in \mathbf{Z}$ is an integer and $\alpha \in [0, 2\pi]$ is the rotation angle. All representations are unitary.
7. All one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}_0^+$ are given by $f(x) = x^k = e^{k\ln(x)}$ with a complex constant $k \in \mathbf{C}$.
8. All unitary one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}_0^+$ are given by $f(x) = x^k = e^{it\ln(x)}$ with a real constant $t \in \mathbf{R}$.
9. All one-dimensional representations $x \mapsto f(x)$ of the group $\mathbf{R}_0$ are given by $f(x) = (\mathrm{sign}x)^\epsilon |x|^k$ with a complex constant $k \in \mathbf{C}$.
10. All one-dimensional representations of $\mathbf{C}_0^1$ are given by

$$f(z) = |z|^k e^{im\arg z}$$

with a complex constant $k \in \mathbf{C}$ and an integer $m \in \mathbf{Z}$.
11. All unitary one-dimensional representations of $\mathbf{C}_0^1$ are given by

$$f(z) = |z|^{it} e^{im\arg z}$$

with a real constant $t \in \mathbf{R}$ and an integer $m \in \mathbf{Z}$.
12. All one-dimensional representations of $\mathbf{C}_0^n$ are given by

$$f(z) = |z_1|^{k_1} \dots |z_n|^{k_n} e^{i(m_1 \arg z_1 + \dots m_n \arg z_n)}$$

with complex constants $k_i \in \mathbf{C}$ and integers $m_i \in \mathbf{Z}$.
13. All unitary one-dimensional representations of $\mathbf{C}_0^n$ are given by

$$f(z) = |z_1|^{it_1} \dots |z_n|^{it_n} e^{i(m_1 \arg z_1 + \dots m_n \arg z_n)}$$

with real constants $t_i \in \mathbf{R}$ and integers $m_i \in \mathbf{Z}$.

A natural way to construct representations is the following: Consider a group G with Haar integral $\int_G f(g)\, dg$. Then we define $L^2(G)$ as usual as the space of all square-integrable functions on G. This space is of course a vector space and seems to be natural to use it as the representation space for representations of G. A simple calculation shows that the mapping $g \mapsto T(g)$ with $T(g)f(h) = f(hg)$ defines indeed a representation of G in $L^2(G)$.

Finally we note that the representation space X together with the group $T(G) = \{T(g) | g \in G\}$ form a transformation group $(X, T(G))$. The orbit of an element $x \in X$ is given by the set $\{T(g)x | g \in G\}$. Sometimes we will call two elements x and $y \in G$ $T(G)$ equivalent if they are members of the same orbit, i.e. if there is a group element $g \in G$ such that $y = T(g)x$. In this case we write also $x \overset{T(G)}{\equiv} y$ or simply $x \equiv y$. $T(G)$ equivalence is an equivalence relation and we find therefore that the space X is the disjoint union of the orbits.

3.3 Exercises

Exercise 3.1 Prove the algebraic representation properties for some of the representations mentioned in the examples 3.1.

Exercise 3.2 Prove theorem 3.5.

Exercise 3.3 Prove some parts of theorem 3.3.

Exercise 3.4 Assume $k(X)$ is a kernel function that depends only on the magnitude of X, i.e. there is a function $h : \mathbf{R}_+ \to \mathbf{R}$ such that $k(X) = h(\|X\|)$.

Define the operator $K : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ by the convolution:

$$Kf(Y) = \int_{\mathbf{R}^n} k(X - Y) f(X)\, dX$$

show that K is an intertwining operator for the rotation group.

The operator given by $h(r) = Ce^{\frac{r^2}{\sigma^2}}$ defines Gaussian blur. In another example h might describe the optical properties of a circular lens, in this case f would describe the intensity distribution of the original object and Kf would be the intensity distribution of the recorded image.