Geometric k Shortest Paths*

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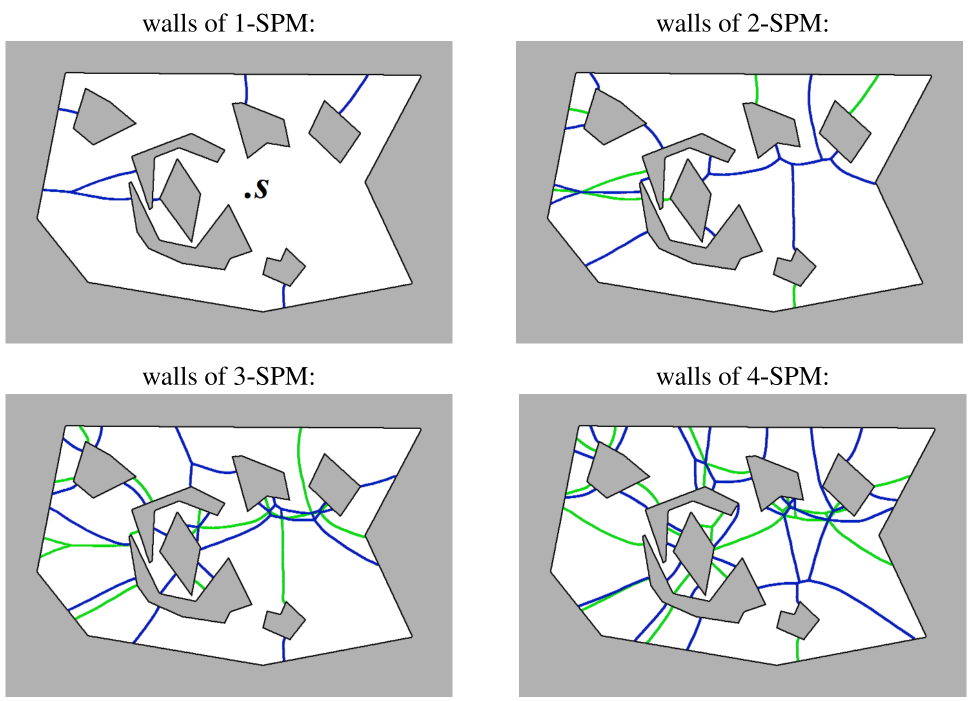
Abstract

We consider the problem of computing k shortest paths in a two-dimensional environment with polygonal obstacles, where the jth path, for $1 \leq j \leq k$, is the shortest path in the free space that is also *homotopically* distinct from each of the first $j-1$ paths. In fact, we consider a more general problem: given a source point s, construct a partition of the free space, called the *kth shortest path map* (k-SPM), in which the homotopy of the k shortest paths in a region has the same structure. Our main combinatorial result establishes a tight bound of $\Theta(k^2h + kn)$ on the worst-case complexity of this map. We also describe an $O((k^3h + k^2n)\log(kn))$ time algorithm for constructing the map. In fact, the algorithm constructs the jth map for every $j \leq k$. Finally, we present a simple visibility-based algorithm for computing the k shortest paths between two fixed points. This algorithm runs in $O(m \log n + k)$ time and uses $O(m + k)$ space, where m is the size of the visibility graph. This latter algorithm can be extended to compute k shortest *simple* (non-self-intersecting) paths, taking $O(k^2m(m + kn)\log(kn))$ time.



We invite the reader to play with our applet demonstrating k-SPMs at
`http://www.cs.helsinki.fi/group/compgeom/kpath_slides/visualize/`.

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1 Introduction

In many applications of mathematical optimization, several "good" solutions are more desirable than a single optimum. This happens because a mathematical model is an imperfect formulation of complex reality, and its various constraints and objectives are only an approximation of the desired goal. Optimization problems are also typically part of a larger system with many interacting parts, where optimal solutions of different parts may be incompatible. In these settings, the system designer must find sub-optimal but high-quality solutions for each part to construct the overall solution. Motivated by these considerations, there is a long history of research on finding k best solutions for discrete optimization problems, including spanning trees and shortest paths in graphs [8, 11, 14, 21].

In this paper, we investigate the fundamental problem of computing k distinct shortest paths among polygonal obstacles in the plane. Because geometric shortest paths live in a continuous (free) space, we need a topological condition on paths to ensure that different paths are non-trivially distinct: otherwise, we can create many nearly identical shortest paths by adding infinitesimal "bumps" to the primary shortest path. The most natural condition is to require paths to have different *homotopy*, where two paths are said to be homotopically equivalent if they can be deformed into each other within the free space of obstacles. Intuitively, two paths are homotopically distinct if they lie on different sides of some obstacle. Multiple shortest paths of distinct homotopies naturally capture the high-level design criteria in geometric environments: e.g., in VLSI design or printed circuit board routing, where obstacles are electronic components, in robot path planning, where obstacles are physical obstructions, in air traffic management, where obstacles model hazardous weather or no-fly zones, etc.

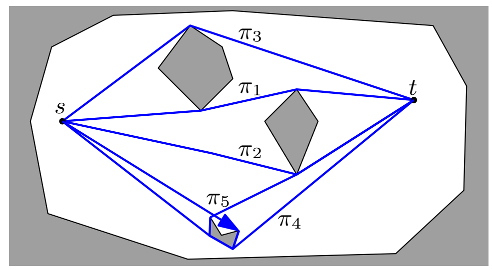


Figure 1: $|\pi_1|<|\pi_2|=|\pi_3|<|\pi_4|<|\pi_5|$. π_1 is the shortest path to t (a 1-path; cf. Def. 2.2), each of π_2 and π_3 is a 2-path, π_4 is a 4-path, π_5 is a 5-path (π_5 is nonsimple—it has a loop going clockwise around the hole).

We consider a more general form of the problem: given a source point s, construct a map of the entire free space, partitioning it into equivalence class regions such that the k shortest paths from s to any point in a single region have the same structure. With this map, one can compute the k shortest paths to any target easily. One of our main results establishes a (tight) bound on the combinatorial complexity of this map; we also give an algorithm for constructing the map. Finally, we consider finding non-self-intersecting kth shortest paths.

Our Results We prove that the edges of the k-SPM are $O(k^2h + kn)$ linear or hyperbolic arcs, and give a construction showing that this bound is tight in the worst case (Section 4). We present an $O((k^3h + k^2n)\log(kn))$ time algorithm (Section 5), using the continuous Dijkstra paradigm, for constructing the map. The algorithm computes the jth shortest path map for all $1 \le j \le k$. Taking this into account, the algorithm is output sensitive: its running time is $O(\log(kn))$ times the total complexity of the first k shortest path maps. By preprocessing the kth map for point-location queries, we can answer kth shortest path queries in $O(\log(kn))$ time; if we want to report (in implicit form) all k shortest paths, the preprocessing time remains the same, but the storage and query time both increase by a factor of k. (If the paths are to be reported explicitly, the complexity of the paths must be added to the query time.) In Section 6, we also present a simpler, albeit asymptotically worse, algorithm for computing the kth shortest path between two fixed points based on the visibility graph. This algorithm runs in $O(m \log n + k)$ time and uses $O(m + k)$ space, where m is the size of the visibility graph. One advantage of this latter algorithm is that it extends easily to find the kth *simple* (non-self-intersecting) path, taking $O(k^2m(m + kn)\log kn)$ time.

Related work Finding shortest paths is also a central problem in the study of graph algorithms. Apart from finding the shortest path itself, considerable attention has been paid to computing its various alternatives including the second, third, and in general kth shortest path between two nodes in a graph; see, e.g., [8, 11] and references therein. On the other hand, *geometric* kth shortest paths have not been explored before. (One problem for which both the graph and the geometric versions were considered is finding the k smallest spanning trees [6, 7].)

In [18] Mitchell surveys many variations of the geometric shortest path problem; for some recent work see [3, 4]. In addition to computing one shortest path to a single target point, a lot of attention in the literature has been devoted to building shortest path *maps*—structures supporting efficient shortest-path queries. A shortest path map can be viewed as the Voronoi diagram of vertices of the domain, where each vertex is (additively) weighted by the shortest-path distance from the source s [12]. Our study of "kth shortest path maps" benefits from notions introduced by Lee [15] for *higher-order* Voronoi diagrams: when bounding the complexity of the maps in Section 4, we employ Lee's ideas to define "old" and "new" features of the map and to derive relationships between them. Higher-order Voronoi diagrams have been recently reexamined in [1, 16, 17, 19]; in particular, [16] considered geodesic diagrams in polygonal domains. Perhaps unsurprisingly, the complexity of our kth shortest path map differs from that of an order-k geodesic Voronoi diagram; the major difference is that homotopies are irrelevant for Voronoi diagrams, but are central in our work.

2 Preliminaries

We are given a closed polygonal domain P with n vertices and h holes; the holes are also called "obstacles" and the domain is called the "free space." We assume that no three vertices of P are collinear and make other general position assumptions below, as needed. We are also given a source point $s \in P$; unless otherwise stated, all paths will have s as an endpoint. For a point $p \in P$, two paths to p are *homotopically equivalent* if one can be continuously deformed to the other while staying within P. Homotopically equivalent paths form an equivalence class (the *homotopy class*) in the set of s–p paths. The unique shortest path in a homotopy class (i.e., a pulled-taut path) is called *locally shortest*.

Observation 2.1. *All bends of a locally shortest path π are at vertices of P and turn toward the corresponding obstacles.*

Let $d(p)$ denote the shortest-path (geodesic) distance from s to p. A vertex v of P is a *predecessor* of p if segment $\overline{vp}$ is in free space and $d(p) = d(v) + |\overline{vp}|$. The *shortest path map* of P (or SPM for short) is the partitioning of P such that all points within the same cell of the SPM have the same unique predecessor. The edges of the partition are called *bisectors*; points on bisectors have more than one predecessor. We distinguish between two types of bisectors: *walls* and *windows*. A bisector is a wall if, for a point p on the bisector, there exist two homotopically different paths to p with length $d(p)$. If there is a unique shortest path to a point p on a bisector, then this bisector is a window; any point p on a window has two predecessors that are collinear with p. We assume that there is a unique shortest path to each vertex of P, and that there are at most three homotopically different shortest paths to each point in P. The former assumption implies that walls are 1-dimensional curves. The endpoints of a wall are either at an obstacle or at a *triple point*, where three walls meet. Windows start at vertices of P and extend until an obstacle or wall is hit. Intuitively, windows can mostly be ignored as far as homotopy types are concerned; walls, by contrast, are central to our study. Fig. "1-SPM" on the title page shows an example of walls in the SPM. By using standard point location structures on the SPM of P, one can query the shortest path length to any point in P in $O(\log n)$ time and, in addition, report the path in linear output sensitive time [12]. Our goal is to compute a similar structure for kth shortest paths.

We now introduce the subject of our study. For a point $p \in P$, let $H(p)$ denote the set of locally shortest paths from s to p of all possible homotopy types.

Definition 2.2. *A path $\pi \in H(p)$ is a k*th shortest path *(or is a k-*path*) to p if there are exactly $k-1$ shorter paths in $H(p)$.*

Figure 1 illustrates the definition. We denote the length of the k-path(s) to p by $d_k(p)$. Notice that, under these definitions, the term 1-path is synonymous with "shortest path" and $d(p) = d_1(p)$.

In order to extend the map concept to k-paths, we first generalize the definition of a predecessor. Let v be an obstacle vertex and i be an integer between 1 and k. For a point p in the plane, the pair (v, i) is a *k-predecessor* of p if the segment $\overline{vp}$ is in free space and $d_k(p) = d_i(v) + |\overline{vp}|$. This implies that a k-path to p can be obtained by concatenating the segment $\overline{vp}$ with the i-path to v. As with the SPM, we assume that each obstacle vertex has a unique i-path for any i, and that there are at most three i-paths in $H(p)$ for each point $p \in P$. Interestingly, for $i > 1$, the former assumption does not follow from a general position assumption. We discuss this issue in Appendix A. For the sake of simplicity, we will ignore the issue in the main body of the paper and stick to the assumption above.

Observe that, given the k-predecessors of all points in the plane and the i-predecessors of all obstacle vertices for $1 \leq i \leq k$, one can construct the k-path to any given point p. The *kth shortest path map* (or k-SPM for short) of P is a subdivision of P into cells such that all points within the same cell have the same unique k-predecessor. In order to construct k-paths from the k-SPM, we also assume that it stores the i-predecessors of all vertices, for all $1 \leq i \leq k$. As with the SPM, one can use standard point location structures to report the k-path length of a query point in $O(\log C_k)$ time, where C_k is the complexity of the k-SPM.

To distinguish the different types of bisectors that form the boundaries of the k-SPM, we generalize the definitions of walls and windows as follows:

Definition 2.3. *A point p is on a k-wall if $H(p)$ contains at least two k-paths.*

Definition 2.4. *A point p is on a k-window if $H(p)$ contains exactly one k-path and p has two k-predecessors.*

Note that the two predecessors of a point p on a k-window must be collinear with p. Furthermore, by the definition of k-paths, a point cannot be on a k-wall and a $(k+1)$-wall at the same time (if a point has two k-paths, then it has no $(k+1)$-path). Similarly to walls in the SPM, k-walls have their endpoints either on obstacles or at triple points, where three k-walls meet. In Section 3, we show that edges of the k-SPM are $(k-1)$-walls, k-walls and k-windows. We also show that our assumption that a k-predecessor is of the form (v, i) with $1 \leq i \leq k$ is indeed correct.

3 Structural results

Consider a path π from s to some target $t \in P$. This path crosses several walls (1-walls, 2-walls, etc.) in P. We define the *crossing sequence* of π as the sequence of positive integers that represents all the i-walls crossed by this path going back from t to s. That is, if π crosses an i-wall, we add i to the sequence. Although it is not strictly necessary, we generally assume an upper bound on the sequence values (the maximum wall class), so that the sequence is finite. We call a sequence a *k-sequence* if it adheres to the following inductive definition:

- A 1-sequence does not contain 1.
- A k-sequence contains $(k-1)$, the first $(k-1)$ occurs before the first k, and the tail of the sequence after the first $(k-1)$ is a $(k-1)$-sequence.

We need the following property of k-sequences, whose proof appears with other omitted proofs in Appendix E.

Lemma 3.1. *A sequence σ cannot be both a k-sequence and an ℓ-sequence if $k \neq \ell$.*

The relation between k-sequences and k-paths is summarized in the following lemma.

Lemma 3.2. *A locally shortest path π is a k-path if and only if its crossing sequence is a k-sequence.*

Proof. We first show that the crossing sequence of a k-path π is a k-sequence. Let us assume that distances have been scaled so that the length of π is 1. Define $p(x)$ for $0 \leq x \leq 1$ as the point on π such that the distance from t to $p(x)$ along π is x. Let $\gamma(x)$ be the subpath of π from $p(x)$ to t. For any $i \geq 1$, let π_i denote the i-path to t ($\pi = \pi_k$). (We assume that t is not on an i-wall, for any $1 \leq i \leq k$.) The concatenation of π_i and $\gamma(x)$ is a path from s to $p(x)$, via t; let $\pi'_i(x)$ denote the shortest path of this homotopy class (Fig. 2, left). All paths $\pi'_i(x)$ must have different homotopy classes for different i.

Let $l_i(x)$ be the length of $\pi'_i(x)$; clearly l_i is continuous. By the definition of k-paths, $l_i(0) \leq l_j(0)$ for $i < j$. On the other hand, $l_k(1) = 0$ and $l_i(1) > 0$ for $i \neq k$. Note that as x grows from 0 to 1, $l_k(x)$ decreases not slower than any other $l_i(x)$, $i \neq k$. Thus, the graph of $l_k(x)$ crosses the graphs of all $l_i(x)$ for $i < k$ exactly once, but no other graphs (Fig. 2, right).

The proof proceeds by induction. A point $p(x)$ is on a j-wall if $l_k(x)$ crosses some other graph at x, and there are exactly $j - 1$ graphs that pass below this intersection. Clearly, if $k = 1$, the path π_k cannot cross a 1-wall, since $l_k(x)$ cannot intersect anything. For $k > 1$, the first intersection of $l_k(x)$ must be with a graph $l_i(x)$ with $i < k$, as described above. This means that $p(x)$ must cross a $(k-1)$-wall before crossing a k-wall. After the $(k-1)$-wall at $x = x^*$, the path $\pi'_k(x^*)$ is the $(k-1)$-path to $p(x)$. By induction, the remainder of the crossing sequence must be a $(k-1)$-sequence.

Finally note that a locally shortest path π must be an i-path for some $i \geq 1$. If the crossing sequence of π is a k-sequence, then it cannot be an i-sequence for $i \neq k$ by Lemma 3.1. Thus $i = k$, and π is a k-path. □

Lemma 3.2 implies that a k-path from s to t crosses walls "in order": it crosses a 1-wall, then a 2-wall, etc., until it crosses a $(k-1)$-wall, after which it reaches t. Also, any prefix of the k-path is an i-path if it crosses the $(i-1)$-wall and not the i-wall. This property of k-paths inspires the construction of a "parking garage" obtained by "stacking" k copies (or *floors*) of P on top of each other and gluing them along i-walls, for $1 \leq i \leq k$. To be precise, the *k-garage* is inductively defined as follows:

> The 1-garage is simply P. The $(k+1)$-garage can be obtained by adding a copy of P (the $(k+1)$-*floor*) on top of the k-garage. We cut both the k-floor of the k-garage and the $(k+1)$-floor along k-walls. We then glue one side of a k-wall on the k-floor to the opposite side of the same k-wall on the $(k+1)$-floor, and vice versa, to obtain the $(k+1)$-garage.

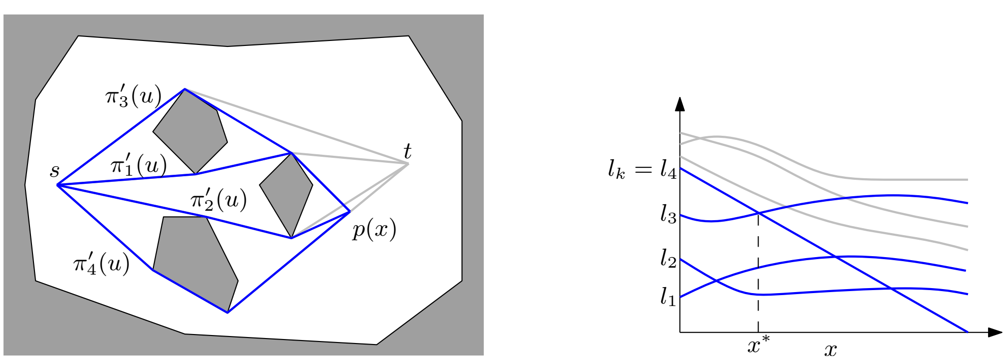


Figure 2: $k = 4$. Left: $\pi'_i(x)$ is the shortest path to $p(x)$, homotopically equivalent to s–π_i–t–$p(x)$. Right: l_k is kth smallest at $x = 0$ and decreases faster than any other l_i.

The k-garage resembles a covering space of P. However, due to the triple points formed by the i-walls ($i < k$), the k-garage is technically not a covering space, but something that is known as a ramified cover. Nonetheless, each path π in the garage can be projected down to a unique path $\pi^{\downarrow}$ in P. The next lemma relates the k-SPM of P to the SPM of the k-garage.

Lemma 3.3. *If π is the shortest path in the k-garage from s on the 1-floor to some t on the k-floor, then $\pi^{\downarrow}$ is a k-path to t.*

Proof. We show that the crossing sequence of $\pi^{\downarrow}$ is a k-sequence. Then, by Lemma 3.2, $\pi^{\downarrow}$ is a k-path. We use the property that every tail of a k-sequence is an i-sequence for some $i \leq k$. If, going back from t to s, π only goes "down" in the k-garage, then it is easy to see that the crossing sequence of $\pi^{\downarrow}$ is a k-sequence. (Because regions on the i-floor are bounded by $(i-1)$- and i-walls, π enters the i-floor by crossing an i-wall and does not cross any i-wall before it exits the i-floor by crossing an $(i-1)$-wall. Thus the tail of π's crossing sequence that starts from any point on the i-floor is an i-sequence.) For the sake of contradiction, assume that π also goes up in the k-garage. Then there must be a point where π goes up to some i-floor, and then goes monotonically down to the 1-floor. The crossing sequence of the corresponding subpath of $\pi^{\downarrow}$ must be of the form $\sigma = (i-1, \sigma_i)$, where σ_i is an i-sequence. If σ is a j-sequence for $j \neq i$, then σ_i must be a j-sequence, which is not possible by Lemma 3.1. If σ is an i-sequence, then σ_i must be an $(i-1)$-sequence, which again is not possible by Lemma 3.1. Finally note that σ must be a j-sequence for some j, since $\pi^{\downarrow}$ is locally shortest. Thus, π only goes down in the k-garage, and the crossing sequence of $\pi^{\downarrow}$ must be a k-sequence. □

Lemma 3.3 directly implies that the SPM on the k-floor of the k-garage is exactly the k-SPM of P. Thus, as claimed before, the edges of the k-SPM consist of $(k-1)$-walls, k-walls, and k-windows. Furthermore, the k-predecessor of a point $p \in P$ must be (v, i) for some i between 1 and k.

4 The complexity of the k-SPM

To obtain an upper bound on the complexity of the k-SPM, we consider a sparser partitioning of P. We define the $(\leq k)$-SPM of P as the partitioning induced by only the k-walls of P. Let $H_k(p)$ be the set of the k shortest homotopy classes to $p \in P$. We refer to $H_k(p)$ as the *k-homotopy set* of p. We would like to claim that the set $H_k(p)$ is constant within each cell of the $(\leq k)$-SPM. Unfortunately we cannot claim this, since the homotopy classes of paths with different endpoints cannot be compared. To overcome this technicality, we define $H_k(p) \oplus \pi$ as the set of homotopy classes obtained by concatenating each path in $H_k(p)$ with π. If π is a path between p and p', then we can directly compare $H_k(p) \oplus \pi$ and $H_k(p')$.

Lemma 4.1. *If p and p' lie in the same cell of the $(\leq k)$-SPM, and π is a path between p and p' that does not cross a k-wall, then $H_k(p) \oplus \pi = H_k(p')$.*

To keep the notation simple, we simply compare $H_k(p)$ and $H_k(p')$ directly, in which case we really mean that we compare $H_k(p) \oplus \pi$ and $H_k(p')$, where π is the shortest path in P between p and p'. Note that π can cross a k-wall. We need the following property of the $(\leq k)$-SPM.

Lemma 4.2. *The cells of the $(\leq k)$-SPM are simply connected.*

We now count the number of k-walls, starting with the case $k = 1$. Let F_1, V_1, and B_1 be the number of faces, triple points, and 1-walls of the (≤ 1)-SPM, respectively. It is easy to see that the (≤ 1)-SPM is simply connected, hence $F_1 = 1$. Now consider the graph G in which each node corresponds to either a hole (including the outer polygon) or a triple point, and there is an edge between two nodes if there is a 1-wall between the corresponding holes/triple points. Since the (≤ 1)-SPM is simply connected, G must be a tree. Hence $B_1 = h + V_1$. (The number of polygons bounding P is $h + 1$.) Furthermore note that the degree of a triple point in G is three, and every node in G has degree at least one. So, by double counting, $2B_1 \geq 3V_1 + h + 1$ or $V_1 \leq h - 1$. To summarize, $F_1 = 1$, $V_1 \leq h - 1$, and $B_1 = h + V_1$.

To bound the complexity of the $(\leq k)$-SPM for $k > 1$, we relate its features to those of the $(\leq k-1)$-SPM. We consider an in-place transformation of the $(\leq k-1)$-SPM into the $(\leq k)$-SPM. We use lower-case letters $a, b, c, \ldots$ to denote the members of $H_k(p)$. Each k-wall of the $(\leq k)$-SPM locally separates regions of P that differ in exactly one of their k shortest path homotopy classes. Note that a k-wall e of the $(\leq k)$-SPM is not present in the $(\leq k+1)$-SPM: if the k-homotopy sets belonging to the two sides of e are $H \cup a$ and $H \cup b$, with $a \neq b$, then the $(k+1)$-homotopy set of points in the neighborhood of e is uniformly $H \cup \{a, b\}$.

The triple points of the $(\leq k)$-SPM fall into two classes, which we call *new* and *old* (borrowing the terms from [15]). If the three k-homotopy sets in the vicinity of a triple point p are $H \cup a$, $H \cup b$, and $H \cup c$, with a, b, and c all distinct, then p is a new triple point. On the other hand, if the three k-homotopy sets are $H \cup \{a, b\}$, $H \cup \{b, c\}$, and $H \cup \{a, c\}$, with a, b, and c all distinct, then p is an old triple point. These names highlight the difference between what happens in the vicinity of p in the $(\leq k+1)$-SPM. If p is a new triple point in the $(\leq k)$-SPM, then it becomes an old triple point in the $(\leq k+1)$-SPM. The three $(k+1)$-walls incident to p in the $(\leq k+1)$-SPM separate points with $(k+1)$-homotopy sets $(H \cup a) \cup b$ from $(H \cup a) \cup c$, $(H \cup b) \cup a$ from $(H \cup b) \cup c$, and $(H \cup c) \cup a$ from $(H \cup c) \cup b$. If p is an old triple point in the $(\leq k)$-SPM, then the $(k+1)$-homotopy set of points in the neighborhood of e is uniformly $H \cup \{a, b, c\}$, and hence p is in the interior of a face of the $(\leq k+1)$-SPM. See Fig. 3.

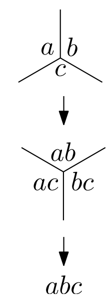


Figure 3: Life cycle of a triple point.

To transform the $(\leq k)$-SPM to the $(\leq k+1)$-SPM, we consider shortest distances to points in each face f of the $(\leq k)$-SPM from its k-walls. The distances from a particular k-wall e are measured according to the homotopy class belonging to the face on the opposite side of e from f. More concretely, let $p \in f$ be a point close to e, and let p' be on the other side of f. Then the shortest paths measured from e use the homotopy class $h_f(e) = H_k(p') \setminus H_k(p)$. For every point $q \in f$, we identify the k-wall e whose homotopy class $h_f(e)$ gives the shortest path to q. Hence $H_{k+1}(q) = H_k(q) \cup h_f(e)$, and this partitions the face f into subfaces, one for each k-wall e, separated by $(k+1)$-walls. To finish the construction of the $(\leq k+1)$-SPM, we erase the k-walls on the boundary of f (recall that their neighborhoods have uniform $(k+1)$-homotopy sets), delete any old triple points whose neighborhoods have uniform $(k+1)$-homotopy sets, and erase any newly added $(k+1)$-walls incident to deleted old triple points on the boundary of f. (These "walls" are actually just windows generated by the triple points; they separate regions with equal $(k+1)$-homotopy sets.)

If a face f of the $(\leq k)$-SPM is bounded by B k-walls, it is initially partitioned into B subfaces. Every pair of subfaces incident to a common old triple point will be merged, so the final number of subfaces is $F' = B - W$, where W is the number of old triple points of the $(\leq k)$-SPM on the boundary of f. Since f is simply connected by Lemma 4.2, and every subface corresponding to a k-wall e must be adjacent to e, the dual graph of the subfaces inside f must be an outerplanar graph. The number of triple points V' added inside f (all of them new) corresponds to the number of (triangular) faces of this outerplanar graph, and hence $0 \leq V' \leq \max(F' - 2, 0)$. By Euler's formula, the number of $(k+1)$-walls created inside f (duals to the edges of the outerplanar graph) is $B' = F' - 1 + V'$.

During the iterative construction of the $(\leq k)$-SPM, we count the features at each step. The description above considers what happens within a single face of the $(\leq k)$-SPM during the transformation to the $(\leq k+1)$-SPM. To account for what happens in all the faces simultaneously, we note that each i-wall is shared between two faces, and each triple point is shared between three faces. Let F_i and B_i be the number of faces and i walls in the $(\leq i)$-SPM. To distinguish between new and old triple points, let V_i and W_i be the number of new and old triple points of the $(\leq i)$-SPM, respectively. (Note that $W_1 = 0$.) If we count just the features added inside faces of $(\leq i)$-SPM, using primed notation, we have

$$\begin{aligned} F'_{i+1} &= 2B_i - 3W_i \\ B'_{i+1} &= 2B_i - 3W_i - F_i + V'_{i+1} \\ V'_{i+1} &\leq 2B_i - 3W_i - 2F_i \\ W'_{i+1} &= 0 \end{aligned}$$

Now let us take into account the deletion of previous i-walls and triple points. All the i-walls and old triple points are deleted between one phase and the next. All new triple points turn into old ones. All subfaces incident to an old triple point merge into one. Thus we obtain the following recurrence relations, whose solution is given by Lemma 4.3.

$$\begin{aligned} F_{i+1} &= F'_{i+1} - B_i + W_i \;=\; B_i - 2W_i & F_1 &= 1 \\ B_{i+1} &= B'_{i+1} \;=\; 2B_i - 3W_i - F_i + V_{i+1} & V_1 &\leq h - 1 \\ V_{i+1} &= V'_{i+1} \;\leq\; 2B_i - 3W_i - 2F_i & B_1 &= h + V_1 \\ W_{i+1} &= V_i & W_1 &= 0 \end{aligned}$$

Lemma 4.3. *The number of faces, walls, and triple points of the* $(\leq k)$*-SPM is* $O(k^2 h)$*.*

We now return to the complexity of the k-SPM. The number of k-walls and $(k-1)$-walls can be bounded by Lemma 4.3. Each k-wall consists of one or more hyperbolic arcs. Note that the number of hyperbolic arcs for a single k-wall is exactly one more than the number of k-windows that end on the k-wall (and a k-window can end on only one k-wall). Hence it is sufficient to count the number of k-windows. Each k-window is an extension of the edge between a vertex v of P and its i-predecessor for $i \leq k$. Thus there can be at most $O(kn)$ k-windows.

Theorem 4.4. *The* k*-SPM of a polygonal domain with* n *vertices and* h *holes has complexity* $O(k^2 h + kn)$*.*

Lower Bound. The bound of Theorem 4.4 is in fact tight. Here we describe an example that has $\Omega(k^2 h)$ k-walls and $\Omega(kn)$ k-windows. See Appendix B for the full details.

Consider the example in Fig. 4, which is constructed so that the shortest paths from s to the vertices p_1, p_2, and p_3 have the same length. Let q be the unique point equidistant from p_1, p_2, p_3. Furthermore, let π_{ij} ($i \in \{1, 2, 3\}$ and $1 \leq j \leq k$) be the j-path from s to p_i, and let l_{ij} be the length of π_{ij}. If the obstacle ω_i is small enough, then π_{ij} simply loops around ω_i zero or more times in a clockwise or counterclockwise direction. Hence, for any $\epsilon > 0$, we can ensure that $|l_{ik} - l_{i1}| \leq \epsilon$ for $i \in \{1, 2, 3\}$ by making the obstacles ω_i small enough. Now define q_{abc} as the unique point such that $|q_{abc} - p_1| + l_{1a} = |q_{abc} - p_2| + l_{2b} = |q_{abc} - p_3| + l_{3c}$. This point must exist, since it is the vertex of an additively weighted Voronoi diagram of p_1, p_2, and p_3. If ϵ is chosen small enough, then q_{abc} must lie in the circle in Fig. 4 for $a, b, c \leq k$.

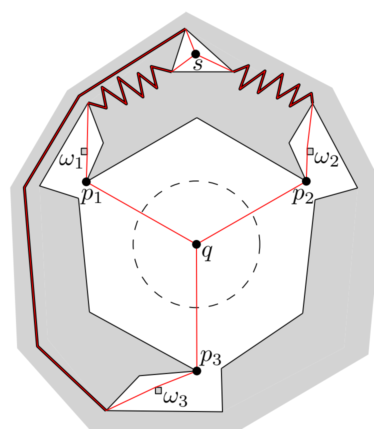
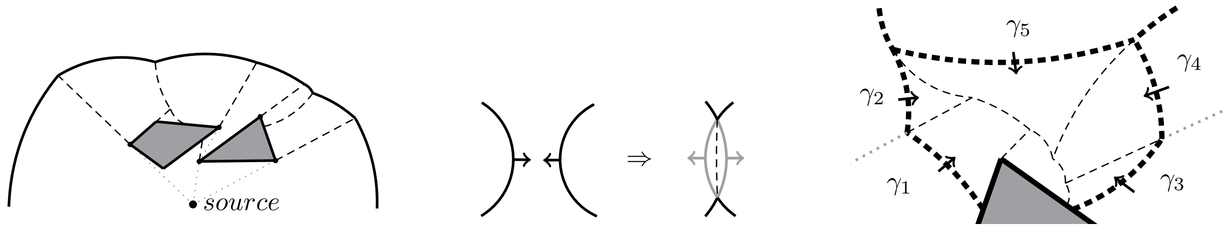


Figure 4: Lower bound gadget.

By construction there are three paths with equal length from s to q_{abc}, and there are exactly $a+b+c-3$ shorter paths from s to q_{abc}. This means that q_{abc} is a triple point of the $(a + b + c - 2)$-SPM. Thus, the number of triple points of the k-SPM is exactly the number of triples (a, b, c) with $1 \leq a, b, c \leq k$ for which $a + b + c - 2 = k$. It is easy to see that there are $\Omega(k^2)$ triples that satisfy these conditions. Note that the gadget has $O(1)$ holes. By connecting $\Theta(h)$ copies of the basic gadget, we get a domain with h holes and $\Omega(k^2 h)$ k-SPM vertices. We can also replace p_3 in one copy by a convex chain of $n' = \Theta(n)$ vertices $v_1, \ldots, v'_n$, such that the line through v_i and v_{i+1} is very close to q for $1 \leq i < n'$. This way each vertex v_i contributes k k-windows to the k-SPM (see Appendix B for details).

Theorem 4.5. *The* k*-SPM of a polygonal domain with* n *vertices and* h *holes can have* $\Omega(k^2 h)$ k*-walls and* $\Omega(kn)$ k*-windows.*



(a) (b) (c)

Figure 5: (a) An expanding wavefront. (b) Two colliding wavelets. After the collision, a wall is formed and both wavelets continue to grow on the next floor. (c) A shortest path map is computed by propagating outside generators into the region R.

5 Computing the *k*-SPM

We now describe how to compute the k-SPM in $O((k^3h + k^2n)\log(kn))$ time. Inspired by the structure of the k-garage and Lemma 3.3, our algorithm iteratively computes the k-SPM for increasing values of k, starting from $k = 1$. Essentially we compute the SPM on the k-garage, one floor at a time. To compute the k-SPM at each iteration, we apply the "continuous Dijkstra" method, which Hershberger and Suri [12] used to compute the shortest path map among polygonal obstacles. We adopt most of the details of the Hershberger–Suri algorithm unchanged, but make a few modifications to support k-SPM computation.

The main idea of the continuous Dijkstra method is to simulate the progress of a wavefront that emerges from the source and expands through the free space with unit speed. If the wavefront reaches a point p at time t, then the shortest path distance between p and the source is t. At any time, the wavefront consists of circular arc *wavelets*, each expanding from a weighted obstacle vertex called a *generator* (see Fig. 5a). A generator γ is represented as a pair (v, w), where v is an obstacle vertex and w is the shortest path distance from the source to v. For a generator $\gamma = (v, w)$ and a point p such that the segment $\overline{vp}$ is contained in free space, the (weighted) distance between γ and p, denoted $d(p, \gamma)$, is defined as $w + |\overline{vp}|$; it is the length of the shortest path from the source to p that passes through v.

Points in the wavelet corresponding to a generator γ at time t satisfy the equation $d(p, \gamma) = t$. We say that a point p is *claimed* by γ if γ is the generator whose wavelet first reaches p; this implies that the shortest path to p passes through v and has length $d(p, \gamma)$. The points where adjacent wavelets on the wavefront meet trace out the bisectors that form the walls and the windows of the shortest path map. Each bisector separates two cells of the shortest path map, each of which consists of points claimed by a particular generator. The bisector curve separating the regions claimed by two generators γ and γ' satisfies the equation $d(p, \gamma) = d(p, \gamma')$. Because $|vp| - |v'p| = w' - w$, the curve is a hyperbolic arc.

Using the continuous Dijkstra approach, the Hershberger–Suri algorithm computes shortest paths from a single source. It also works for shortest paths from multiple sources with delays. This is summarized in the following lemma, which was proved in [12].

Lemma 5.1 ([12]). *Given a set of polygonal obstacles with n vertices and a set of $O(n)$ sources with delays, one can compute the corresponding shortest path map in $O(n \log n)$ time.*

To compute the k-SPM, we apply the continuous Dijkstra framework on each floor of the k-garage. Imagine that we start a wavefront expansion from the source. When a wavelet collides with another wavelet during propagation (and thus forms a 1-wall), the portion of the wavelet that is claimed by the other wavelet continues to expand on the 2-floor (see Fig. 5b). Since this portion of the wavelet has passed through a 1-wall, it represents a set of 2-paths, by Lemma 3.3. Any bisectors formed by adjacent wavelets on the 2-floor belong to the 2-SPM. Similarly to the 1-floor, when two wavelets collide on the 2-floor, they form a 2-wall and continue to expand on the 3-floor. We continue to push the colliding wavelets up to higher floors until they reach the k-floor, which will correspond to the k-SPM.

Notice that the wavefront expansion on a single floor is not affected by the expansion on other floors, with the exception of wavelet collisions on the previous floor. We now describe a method that exploits this fact to compute the k-SPM once the $(k-1)$-SPM has been computed. Thus we can construct the k-SPM by first running the Hershberger–Suri algorithm to compute the 1-SPM and then iteratively applying this step to compute higher floor SPMs.

We compute the k-SPM from the $(k-1)$-SPM as follows. The boundaries of the $(k-1)$-SPM are formed by $(k-1)$-windows, $(k-1)$-walls and $(k-2)$-walls. The $(k-1)$-windows and $(k-2)$-walls do not appear in the k-SPM, so we simply remove them from the map. The $(k-1)$-walls remain in the map and they subdivide the free space into simply connected regions (by Lemma 4.2). To complete the k-SPM, in each such region we compute a special shortest path map whose walls and windows form the k-windows and k-walls of the k-SPM.

The shortest path map computed in each region R is drawn with respect to multiple "restricted" sources with delays, which are determined as follows. Consider a $(k-1)$-wall W bounding R in the $(k-1)$-SPM and let $\gamma = (v, w)$ be the generator that claims the region outside R in the vicinity of W. (It is possible that both sides of W are contained in R. In this case, our description applies to the generators claiming both sides.) Note that W is formed by the collision of the wavelet of γ with another wavelet, and the wavelet of γ is pushed up to the k-floor inside R. Conceptually, we want to continue expanding the wavelet of γ inside R. To do this, we introduce γ as a source at v with delay w and impose the additional restriction that all paths from γ to the interior of R pass through W.[1] In other words, we do not allow any paths from v that do not pass through W. We create sources in this manner for each $(k-1)$-wall bounding R and draw the shortest path map with respect to these sources (see Fig. 5c).

We can compute the shortest path map inside each region by running a single instance of the Hershberger–Suri algorithm for delayed sources. Our restrictions necessitate some modifications, described in Appendix C, but with these modifications the algorithm computes the shortest path map in each region bounded by $(k-1)$-walls. Since the paths used to compute the map in each region are k-paths by Lemma 3.3, the walls and windows of the map form the k-walls and k-windows of the k-SPM. This completes the construction of the k-SPM.

Theorem 5.2. *Given a source point in a polygonal domain with n vertices and h holes, the corresponding k-SPM can be computed in $O((k^3h+k^2n)\log(kn))$ time. If the total complexity of all i-SPMs for $1 \le i \le k$ is M, then the running time is $O(M\log(kn))$.*

6 Visibility-based algorithms

The k-SPM provides an efficient data structure for querying k-paths from a fixed source s. If we are simply interested in the k-path between two fixed points s and t, then it may be inefficient to construct the k-SPM for large values of k. In this section we present a simple visibility-based algorithm to compute the k-path between s and t. For large k, this algorithm is faster than the k-SPM approach. Moreover, this algorithm is relatively easy to implement and may therefore be of more practical interest.

We first compute the visibility graph (VG) of P in $O(n\log n + m)$ time [9, 20], where $m = O(n^2)$ is the size of VG. We also include visibility edges to s and t. The graph contains every locally shortest path from s to t and hence also the k-path to t. However, we cannot simply compute the kth shortest path in VG, since different paths in the graph may be homotopic. We therefore modify VG so that locally shortest paths are in one-to-one correspondence with paths in the modified graph—this ensures that different paths in the graph belong to different homotopy classes. First, we make the graph directed by doubling each edge. Then we expand each vertex v as illustrated in Fig. 6: Draw the two lines supporting the two obstacle edges incident

[1] We also require that the subpath between v and W is a straight line.

Figure 6: Vertex expansion for the taut graph.

to v; the lines partition the relevant visibility edges at v into two sets A and B (the visibility edges between the lines opposite the obstacle are irrelevant, because they cannot be used by shortest paths). Radially sweep a line through v, initially aligned with one of the obstacle edges, until it is aligned with the other obstacle edge. For each visibility edge e encountered, create a node with an incoming edge if $e \in A$, and an outgoing edge if $e \in B$. Connect all created nodes with a directed path. Also make a copy of this construction with all edges reversed. The expansion of v is connected with other expansions in the obvious way, as dictated by the visibility graph. Finally, remove edges directed toward s and away from t. The constructed graph—which we call the *taut graph* $\vec{G}(P)$—has $O(m)$ vertices and $O(m)$ edges and can be built in $O(m)$ time. Note that, by construction, every path in $\vec{G}(P)$ must be locally shortest and every locally shortest path from s to t exists in $\vec{G}(P)$.

We can now use the algorithm by Eppstein [8] to compute the kth shortest path from s to t in $\vec{G}(P)$, which corresponds to the k-path from s to t in P.

Theorem 6.1. *The k-path between s and t in P can be computed in $O(m \log n + k)$ time, where m is the size of the visibility graph of P.*

Interestingly, this approach can be extended to compute the kth shortest *simple* path (*simple k-path*) between s and t in polynomial time. Here we define a *simple path* as a path that does not cross itself, although repeated vertices and segments are allowed. To compute simple k-paths, we adapt Yen's algorithm [21] for computing simple k-paths in directed graphs (here "simple" means free of repeated nodes). The details are non-trivial and can be found in Appendix D. We obtain the following result.

Theorem 6.2. *The simple k-path between s and t can be computed in $O(k^2 m(m + kn) \log kn)$ time, where m is the number of edges of the visibility graph of P.*

7 Concluding remarks

We have introduced the k-SPM, a data structure that can efficiently answer k-path queries. We provided a tight bound for the complexity of the k-SPM, and presented an algorithm to compute the k-SPM efficiently. Our algorithm simultaneously computes all the i-SPMs for $i \leq k$. Whether there is a more direct algorithm to compute the k-SPM is an interesting open problem. We also provided a simple visibility-based algorithm to compute k-paths, which may be of practical interest, and is more efficient for large values of k. This latter approach can be extended to compute simple k-paths. Unfortunately, we do not know how to extend the k-SPM to simple k-paths. It seems that simple k-paths lack the useful property that a subpath of a simple k-path is a simple i-path for $i \leq k$. This makes finding a more efficient algorithm to compute simple k-paths a challenging open problem.

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A Handling Degeneracies and Tie-Breaking

For simplicity of analysis we assumed that P satisfies the following conditions:

1. No three of the vertices of P, including the source s, are collinear.

2. There are at most three homotopically different i-paths to a single point in P, for $1 \leq i \leq k$. Equivalently, no four i-walls meet at a single point.

3. There is a unique i-path to each vertex of P, for $1 \leq i \leq k$. Equivalently, no i-wall goes through a vertex of P.

With these assumptions all walls are one-dimensional curves that meet only at triple points.

We now describe briefly how to adapt our analysis if these assumptions are false. If we are dealing with first shortest paths only, then we can simply apply the standard technique of (symbolic) perturbation to the input (i.e., perturb the positions of the vertices) so that the input is in general position and satisfies all of the assumptions. However, for k-paths with $k \geq 2$, we need more than perturbation to enforce all assumptions. In particular, Assumption 3 cannot be enforced by perturbation because it can be violated even when the input is non-degenerate. For an example see Fig. 7: The 1-path from s to v is a straight line. There are two 2-paths from s to v, labeled π_1 and π_2. The paths π_1 and π_2 are homotopically different; they pass through v first and then loop around the same obstacle in different directions to return to v. Both π_1 and π_2 have the same length, and thus v is on the 2-wall. This implies that v and all of the points to its left below ray r have two distinct 2-paths and thus belong to a 2-wall; the 2-wall is thus a region, not a curve.

In order to avoid this issue, we introduce a tie-breaking mechanism between the paths so that all paths to an obstacle vertex are strictly ordered by length and thus each obstacle vertex has a unique i-path. In particular, suppose that π_1 and π_2 are two i-paths from s to a vertex v with the same length. We break the tie between π_1 and π_2 by arbitrarily assuming that one of the two paths is infinitesimally shorter than the other. Conceptually, this mechanism perturbs the i-wall by moving it slightly to one side. As a result, the i-wall does not go through v and Assumption 3 is satisfied. Once the tie is broken, we assume that all paths that are obtained by extending π_1 and π_2 with the same subpath preserve this order, maintaining consistency.[2]

By applying (symbolic) perturbation and enforcing a strict virtual order between the paths via tie-breaking, we guarantee all our assumptions.

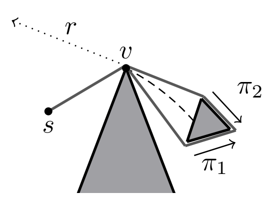


Figure 7: The equal-length paths π_1 and π_2 are both 2-paths to v. The 2-wall is shown with a dashed line.

[2] This still applies even if there are other tie-breakings in the extending subpath.

B Lower Bound

The construction of the lower bound example has already been explained in the main body of the paper. Recall that a point q_{abc} (for $1 \leq a, b, c \leq k$) is the unique point that satisfies $|q_{abc} - p_1| + l_{1a} = |q_{abc} - p_2| + l_{2b} = |q_{abc} - p_3| + l_{3c}$, and $l_{ik} - l_{i1} \leq \epsilon$, for any $1 \leq i \leq 3$. We first need to show that all points q_{abc} lie in the circle shown in Fig. 4, if ϵ is chosen small enough.

Lemma B.1. *If $\epsilon < |q - p_i|$ for $i \in \{1, 2, 3\}$, then $|q_{abc} - q| < \epsilon$, for $a, b, c \leq k$.*

Proof. Points p_1, p_2, and p_3 are the vertices of an equilateral triangle, with q at its center. Define $L = |q - p_1|$. By assumption, $L > \epsilon$. Since $0 \leq l_{ij} - l_{i1} \leq \epsilon$, for $i \in \{1, 2, 3\}$ and any $1 \leq j \leq k$, and

$$|q_{abc} - p_1| + l_{1a} = |q_{abc} - p_2| + l_{2b} = |q_{abc} - p_3| + l_{3c},$$

we have $|q_{abc} - p_i| \leq |q_{abc} - p_j| + \epsilon$ for any $1 \leq i, j \leq 3$. The locus of points satisfying these inequalities is bounded by six hyperbolic arcs, as shown in Fig. 8. Each arc bulges toward the center, so putting q_{abc} at a vertex of the region maximizes $|q_{abc} - q|$. There are two classes of vertices of the region. They are defined by intersections of hyperbolae arranged in three pairs along the three angle bisectors at p_1, p_2, and p_3. By symmetry we can solve for points lying on an angle bisector satisfying the difference relations shown in Fig. 8. We apply the law of cosines to find minimum and maximum values of d, the distance from any of the p_i to the intersections of hyperbolae on the angle bisector at p_i. Solving for the lower bound on d (Fig. 8(left)), we have

$$\begin{aligned}
d^2 + 3L^2 - 2d\sqrt{3}L \cos\frac{\pi}{6} &= (d + \epsilon)^2 \\
3L^2 - 3dL &= 2d\epsilon + \epsilon^2 \\
d = \frac{3L^2 - \epsilon^2}{3L + 2\epsilon} &= L - \frac{2}{3}\epsilon + \frac{\epsilon^2}{3(3L + 2\epsilon)} \\
&> L - \frac{2}{3}\epsilon.
\end{aligned}$$

Solving for the upper bound (Fig. 8(right)), we have

$$\begin{aligned}
d^2 + 3L^2 - 2d\sqrt{3}L \cos\frac{\pi}{6} &= (d - \epsilon)^2 \\
3L^2 - 3dL &= -2d\epsilon + \epsilon^2 \\
d = \frac{3L^2 - \epsilon^2}{3L - 2\epsilon} &= L + \frac{2}{3}\epsilon + \frac{\epsilon^2}{3(3L - 2\epsilon)} \\
&< L + \epsilon
\end{aligned}$$

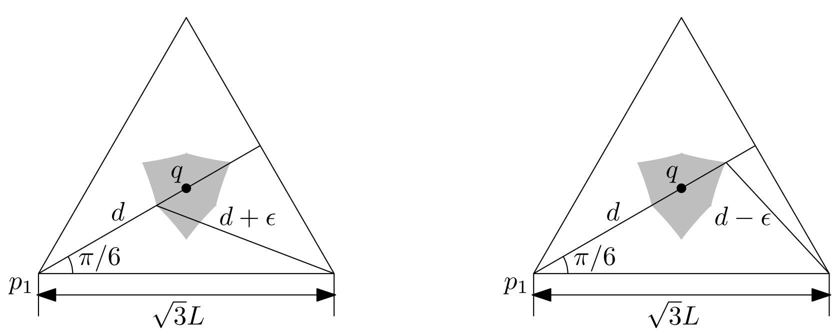


Figure 8: Extreme locations of q_{abc}.

since $L > \epsilon$. Because q_{abc} is constrained to lie in this hyperbolically bounded region, and the maximum distance from q to the boundary of the region is less than ϵ, we have $|q_{abc} - q| < \epsilon$. □

As argued in the main body of the paper, a point q_{abc} is a triple point in the k-SPM if $a + b + c - 2 = k$, and there are $\Omega(k^2)$ such triples. The gadget in Fig. 4 has a constant number of holes. To obtain a lower bound of $\Omega(k^2 h)$ k-walls, we need to connect h copies of the gadget together. We can do this as follows. First construct a thin polygon in the shape of a star graph with h leaves. Then connect a copy of the gadget to each of the leaves by opening up the gadget at the region that contains the source of the gadget (and scaling the gadgets so that they do not overlap). Finally we place the source s at the center of the star. This results in a polygonal domain with $\Theta(h)$ holes for which the k-SPM contains $\Omega(k^2 h)$ triple points. Since triple points are adjacent to three k-walls, this directly implies that there must also be $\Omega(k^2 h)$ k-walls.

In order to extend the construction to have $\Omega(kn)$ k-windows as well, we replace the vertex p_3 in one of the gadgets by a convex chain of $\Theta(n)$ vertices, as explained in the main body of the paper. We then obtain the following result.

Theorem 4.5. *The k-SPM of a polygonal domain with n vertices and h holes can have $\Omega(k^2 h)$ k-walls and $\Omega(kn)$ k-windows.*

Proof. We use the construction described above. This means that the number of triple points is $\Omega(k^2 h)$. However, the points q_{abc} might coincide for different values of a, b, and c. To argue that this is not the case, we rewrite the equations for q_{abc} as follows:

$$\begin{aligned} |q_{abc} - p_1| - |q_{abc} - p_2| &= l_{2b} - l_{1a} \\ |q_{abc} - p_1| - |q_{abc} - p_3| &= l_{3c} - l_{1a} \end{aligned}$$

A single one of these equations describes a hyperbolic arc. Also, if $l_{2b} - l_{1a}$ differs for different values of a and b, then the corresponding hyperbolic arcs are disjoint. The same holds for the second equation. That means that $q_{abc} = q_{a'b'c'}$ if and only if $l_{2b} - l_{1a} = l_{2b'} - l_{1a'}$ and $l_{3c} - l_{1a} = l_{3c'} - l_{1a'}$. We assume that all obstacles in the gadget have the same size, so that l_{ij} depends only on j. Hence, the location of q_{abc} depends only on the differences among a, b, and c. Finally note that for triples a, b, c with $a + b + c - 2 = k$, the differences among a, b, and c are unique. Thus, all triple points q_{abc} on the k-SPM are unique. (It does not matter that $q_{abc} = q_{(a+1)(b+1)(c+1)}$, since they are not part of the same map.)

Next we need to show that the k-SPM can have $\Omega(kn)$ k-windows. Since the number of vertices in the convex chain at p_3 is $\Theta(n)$, it is sufficient to show that each vertex in the chain (except the first) contributes k k-windows to the k-SPM. Let e_j be the edge formed by extending the edge between v_j and v_{j+1} toward q until it hits the boundary of P. We claim that, for every $i \leq k$, there must be a point $t \in e_j$ such that the path π consisting of the i-path to v_j followed by the segment $\overline{v_j t}$ is the k-path from s to t. In other words, t is on a k-window. If t is at v_j, then π is an i-path by definition. If t is the other endpoint of e_j and e_j is sufficiently close to q, then π must be an ℓ-path for $\ell > k$. Lemma 3.2 now implies that there must be a $t \in e_j$ such that π is the k-path from s to t. Thus, each vertex in the convex chain (except the first) contributes k k-windows, and the k-SPM has $\Omega(kn)$ k-windows. □

C Implementing the Continuous Dijkstra Algorithm

The Hershberger–Suri algorithm [12] for finding shortest paths among polygonal obstacles simulates the wavefront expansion on a "conforming subdivision" of the free space. Each internal (free-space) edge e of this subdivision is contained in a set of cells whose union is called the "well-covering region" of e and denoted by $\mathcal{U}(e)$. (See Fig. 9a.) Briefly, the wavefront simulation computes the wavefront passing through each internal subdivision edge. The wavefront for a subdivision edge e is computed by propagating and combining the already computed wavefronts on the edges bounding $\mathcal{U}(e)$.[3] Once the wavefronts for all edges have been computed, the shortest path map in each subdivision cell is constructed locally by computing a weighted Voronoi diagram for the generators that claim the boundaries of the cell or are inside the cell. These cell-wide maps are then easily combined into a global shortest path map.

As sketched in Section 5, we apply the Hershberger–Suri algorithm within regions of free space bounded by $(k-1)$-walls to compute the k-SPM. This requires two extensions to the algorithm:

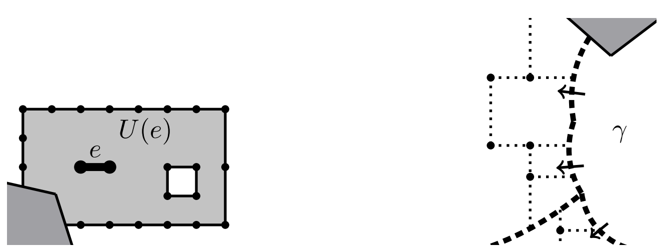
First, in order to divide the free space into the separate regions of interest, we treat the $(k-1)$-walls as obstacles. The algorithm from [12] that builds the conforming subdivision of the free space assumes that the obstacles have straight boundaries, which may not hold for the $(k-1)$-walls. (Each $(k-1)$-wall consists of hyperbolic arcs.) We overcome this issue by using a slightly modified algorithm that creates conforming subdivisions for "curved" obstacles (within the same complexity bounds). This modified algorithm was described in [13], where it was used to compute shortest paths among curved obstacles; we omit its details. Note that even though we are using a subdivision that may have curved edges, we still apply the wavefront propagation algorithm for polygons on this subdivision, because each curved edge resides on a $(k-1)$-wall whose claiming generator is already known. Thus, the curved edges do not take part in the wavefront propagation or yield additional generators, as they do in [13].

Our second modification to the shortest path algorithm is the initialization of wavefront propagation in the subdivision. The original algorithm of Hershberger and Suri starts the propagation by passing the wavefront directly from each source point s to all edges e whose well covering region $\mathcal{U}(e)$ contains s. The sources in our setting are generators to be propagated into regions, each through its own $(k-1)$-wall, and thus we need a different way to initialize the wavefront. To meet our requirements, we initiate the wavefront propagation in the vicinity of the $(k-1)$-walls rather than the generators. In particular, the wavefront for a single generator γ is directly propagated to

(1) All edges e that bound a cell into which γ is to be propagated through a $(k-1)$-wall (see Fig. 9b).
(2) All edges e such that e contains an edge from (1) in its well-covering region $\mathcal{U}(e)$.

Note that propagating a generator's wavefront to an edge does not mean that the wavefront claims the edge, because some or all of the wavefront may be eliminated by other propagated wavefronts when they are merged to compute the final wavefront.

[3] Well covering regions have special properties ensuring an acyclic propagation order between the edges of the subdivision.



(a) (b)

Figure 9: (a) The well-covering region $\mathcal{U}(e)$ (light gray) for an edge e in the conforming subdivision. (b) The set of subdivision edges in the vicinity of the $(k-1)$-walls through which a generator γ is propagated.

These two modifications enable the algorithm to compute the wavefront passing through every edge in the conforming subdivision, and hence to find the SPM in each region bounded by $(k-1)$-walls. The union of these shortest path maps is the k-SPM.

Theorem 5.2. *Given a source point in a polygonal domain with n vertices and h holes, the corresponding k-SPM can be computed in $O((k^3h+k^2n)\log(kn))$ time. If the total complexity of all i-SPMs for $1 \le i \le k$ is M, then the running time is $O(M\log(kn))$.*

Proof. We construct the k-SPM iteratively for increasing values of k as described. We argue that at each iteration, the time spent to construct the k-SPM from a given $(k-1)$-SPM is $O((k^2h+kn)\log(kn))$. This implies the total time spent is $O((k^3h+k^2n)\log(kn))$.

Define the complexity of the i-SPM to be M_i. By Theorem 4.4, $M_{k-1} = O(k^2h+kn)$. We construct the k-SPM by running the modified Hershberger–Suri algorithm as described above. The algorithm is run on a set of obstacles with $O(M_{k-1})$ vertices (including the original obstacle vertices and the endpoints of the hyperbolic arcs forming the $(k-1)$-walls) with $O(M_{k-1})$ delayed sources (at most two sources per hyperbolic arc). By Lemma 5.1 (which applies also to our modified algorithm), the algorithm completes in $O(M_{k-1}\log M_{k-1}) = O(M_{k-1}\log(kn))$. The total complexity of all i-SPMs, for $1 \le i \le k$, is $\sum_{i=1}^{k} M_i = M$, and so the total running time is $O(M\log(kn))$ This completes the proof. □

D Simple k-paths

Our definition of k-paths allows a path to be self-crossing. This may be undesirable for many applications. In this section we show how to compute the kth shortest *simple* path (*simple k-path*) in polynomial time, albeit slower than when we allow self-crossing paths. Here we define a *simple path* as a path that does not cross itself, although repeated vertices and segments are allowed. Note that we cannot use one of our previous methods to solve this problem: the simple 3-path may be a k-path for arbitrarily high k.

To compute the simple k-path between two fixed points s and t in P, we need to treat s and t as point obstacles (otherwise pulling a path taut may introduce self-crossings), but this trivializes the problem (the path may wind around s or t for free) unless special restrictions are added; therefore, for ease of presentation, we limit our attention to the case in which s and t are located on the boundaries of obstacles.

We again use the taut graph $\vec{G}(P)$ to reduce the problem to a graph problem. The taut graph ensures that every path between s and t is locally shortest, but it still allows crossings. To avoid crossings, we adapt Yen's algorithm [21] for simple k-paths in directed graphs (in graphs, "simple" means free of repeated nodes). Yen's algorithm first computes the shortest path, which must be simple; the same is true in our geometric setting. Next, the algorithm "expands" the shortest path π in the following way: It considers every possible prefix of π and chooses a next edge e that is different from the next edge in π. It then finds the shortest path starting from the endpoint of e that avoids the prefix including e; this ensures that the resulting path is simple and different from π. Such paths are computed for every possible prefix and edge e, and stored in a heap; the shortest such path in this heap is the simple 2-path. The algorithm continues by expanding the simple 2-path and repeats this process, selecting the shortest of all the expanded paths in the heap, until the simple k-path is found.

Note that we cannot use Yen's algorithm directly on $\vec{G}(P)$, since a simple path in $\vec{G}(P)$ is not necessarily simple in the geometric sense. To make this algorithm work in our setting, we need to make a small modification. Before we compute the shortest path with a given prefix π_p (including e), we add π_p as an obstacle to P, obtaining a new polygon P'. We then work with the taut graph $\vec{G}(P')$ of the new polygon (we separate each vertex of π_p and the corresponding obstacle vertex by an infinitesimal amount to allow paths that abut π_p but do not cross it). We need to show that the locally shortest path with a given prefix, i.e., the shortest path in $\vec{G}(P')$ starting after e, is simple. Clearly π_p is simple, and the suffix cannot cross π_p, but it is not clear that the suffix itself is simple, especially given the geometric nature of our paths. In order to prove this, we need some additional results.

Let π_{pq} denote the subpath of a path π between two points $p, q \in \pi$. We can apply a *shortcut* to a path π by replacing π_{pq} by the straight segment $\overline{pq}$, so long as $\overline{pq}$ lies in free space. A shortcut is *valid* if it does not change the homotopy class of the path. We assume that a valid shortcut $\overline{pq}$ does not cross π_{pq}, for otherwise we can cut up the shortcut into multiple smaller shortcuts. A shortcut is valid if and only if the cycle formed by π_{pq} and $\overline{pq}$ does not contain an obstacle. Note that a locally shortest path has no valid shortcuts. Furthermore, we can make a path locally shortest by repeatedly applying valid shortcuts until no more valid shortcuts exist.

A path π is *x-monotone* if every vertical line crosses π only once. Given a path π in P, we can obtain π' by repeatedly applying valid vertical shortcuts to π until no more valid vertical shortcuts exist. We call π' the *vertical reduction* of π. We can then find the smallest set S of vertices of P along π' such that the subpath of π' between two adjacent (along π') vertices in S is x-monotone. We call the vertices in S the *extremal vertices* of π'.

Now consider two homotopic paths π_1 and π_2 and their vertical reductions π_1' and π_2'. As was shown in [2, Lemmas 1 and 7], the set of extremal vertices of π_1' and π_2' must be the same. Hence the set of extremal vertices depends only on the homotopy class of π_1, and we can also speak of the extremal vertices of π_1. Finally note that a locally shortest path is its own vertical reduction. Thus the locally shortest path homotopic to a path π must pass through the extremal vertices of π.

We can now prove the following result.

Lemma D.1. *The shortest path in $\vec{G}(P')$ that starts with a fixed (simple) prefix π_p must be simple in P.*

Proof. For the sake of contradiction, assume that the shortest path π with fixed prefix π_p crosses itself at the point $x \in \pi$ on edge e^*, where e^* is the first crossing edge after π_p. (See Fig. 10a.) Assume w.l.o.g. that the bend at the vertex v before e^* makes a right turn. We can rotate the polygonal domain so that the direction of e^* is infinitesimally clockwise from vertically up. As a result, v is an extremal vertex of π.

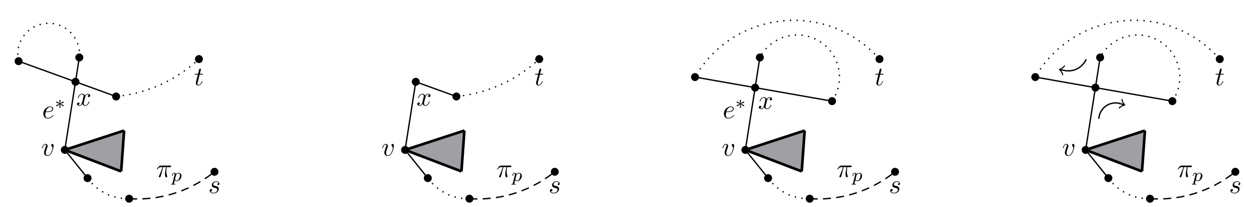
We will show that there is a locally shortest path π' that is shorter than π and also makes a right turn at v. Since a locally shortest path must turn toward obstacles, it is sufficient to show that π' is shorter and passes through v. We first construct a path π'' that is not longer than π, and then let π' be the locally shortest path homotopic to π'', which is shorter than π.

The path π (from s to t) crosses e^* either (i) from left to right (as in Fig. 10a) or (ii) from right to left (as in Fig. 10c). Let π^* be the subpath of π between the two occurrences of the crossing. In case (i) π'' is obtained by eliminating π^*. (See Fig. 10b.) In case (ii) π'' is obtained by reversing π^*. (See Fig. 10d.) In case (i) π'' is clearly shorter than π. In case (ii) π'' has the same length as π, but note that π' must then be shorter.

In both cases π'' makes a right turn at x. Now note that every vertical shortcut of π'' must also exist in π. To see that, notice that the only shortcuts of π' we need to consider are those that span π^* in case (i) or span or touch π^* in case (ii); any other shortcut also exists in π. A vertical shortcut that connects any point before π^* to a point on or after π^* is blocked by v (i.e., the shortcut is not valid). A shortcut of π' within π^* must also exist in π. A shortcut from a point on π^* to point after π^* (in case (ii)) is blocked by the first extremal vertex after π^*. Since every vertical shortcut of π'' exists in π and π is locally shortest (i.e. has no valid shortcuts), π'' must be its own vertical reduction. Thus, v is an extremal vertex of π'', and π' must pass through v.

Finally we need to show that π' is actually a path in $\vec{G}(P')$. Note that $\vec{G}(P')$ contains all locally shortest paths in P that do not cross the fixed prefix π_p. So it is sufficient to show that π' does not cross π_p. Since π did not cross π_p, the same is true for π''. We can obtain π' from π'' by repeatedly applying valid shortcuts. It is now sufficient to show that any valid shortcut $\overline{pq}$ between $p, q \in \pi''$ cannot cross π_p. For the sake of contradiction, assume that $\overline{pq}$ crosses π_p. That means that some part of π_p must go inside the cycle C formed by $\overline{pq}$ and π''_{pq}. Note that s is outside C since we assumed that s belongs to an obstacle. If π_p ends inside C, then there must be an obstacle inside C, which means that the shortcut was not valid. Otherwise, π_p must also leave C. It cannot leave through π''_{pq}, since π'' did not cross π_p. If it leaves C through $\overline{pq}$, then there must be a bend inside C. But this again means that there is an obstacle inside C, which contradicts the validity of the shortcut.

Thus, the path π' contains π_p, it exists in $\vec{G}(P')$, and it is shorter than π. This contradicts the choice of π. $\square$



(a) (b) (c) (d)

Figure 10: (a) π crosses e^* from left to right. (b) π'' is obtained by eliminating π^*. (c) π crosses e^* from right to left. (d) π'' is obtained by reversing π^*.

Thus, if we compute $\vec{G}(P')$ before every shortest path computation, every path obtained by our adaptation of Yen's algorithm must be simple. We finally obtain the following result.

Theorem 6.2. *The simple k-path between s and t can be computed in $O(k^2 m(m+kn)\log kn)$ time, where m is the number of edges of the visibility graph of P.*

Proof. The simple k-path has at most kn edges since each vertex of P can be visited at most k times. This means that a simple k-path can have at most $O(km)$ prefixes (including e). To compute $\vec{G}(P')$, note that every visibility edge of P' is also a visibility edge of P, although some edges may occur multiple times in P' (edges of P in the prefix are duplicated). Hence, to compute P', we need to understand which visibility edges of P still exist in P'. By considering the prefixes in order of increasing length (one edge at a time), we only need to check which visibility edges of P cross the last edge of the prefix, which can be computed in $O(m)$ time per prefix. Since the prefix can have at most kn edges, the visibility graph of P' can have at most $O(m+kn)$ edges. We can then compute $\vec{G}(P')$ in $O(m+kn)$ time. Finally, we can use Dijkstra's algorithm [5] to compute the shortest path in $\vec{G}(P')$ after the prefix in $O((m+kn)\log kn)$ time. To obtain the simple k-path, we need to expand $k-1$ paths. Each path may have $O(km)$ prefixes, and the shortest path for each prefix can be computed in $O((m+kn)\log kn)$ time. Thus, we can compute the simple k-path in $O(k^2 m(m+kn)\log kn)$ time. □

E Omitted Proofs

Lemma 3.1. *A sequence σ cannot be both a k-sequence and an ℓ-sequence if $k \neq \ell$.*

Proof. Assume without loss of generality that $\ell < k$. The definition of a k-sequence directly implies the following properties: (i) A k-sequence contains all integers in $\{1, \dots, k-1\}$, and (ii) every tail of a k-sequence is an i-sequence for some $i \leq k$.

Let k be the smallest number for which the lemma does not hold; clearly $k > 1$. If $\ell = 1$, then σ does not contain 1 while a k-sequence must contain 1 (property (i)); so assume $\ell > 1$. Since $k > \ell$, σ must contain ℓ (property (i) again). By definition, the tail of σ after one of the occurrences of ℓ is an ℓ-sequence. Since σ is also an ℓ-sequence, it must contain $(\ell - 1)$ before ℓ, and the tail of σ after $(\ell - 1)$ is an $(\ell - 1)$-sequence. In particular, the tail of σ after the occurrence of ℓ mentioned above must also be an i-sequence for some $i \leq \ell - 1$ (property (ii)). But then the lemma does not hold for $k = \ell, \ell = i$, contradicting our choice of k. □

Lemma 4.1. *If p and p' lie in the same cell of the $(\leq k)$-SPM, and π is a path between p and p' that does not cross a k-wall, then $H_k(p) \oplus \pi = H_k(p')$.*

Proof. We reuse ideas from the proof of Lemma 3.2. Let us assume that distances have been scaled so that the length of π is 1. Define $p(x)$ $(0 \leq x \leq 1)$ as the point on π such that the distance from p to $p(x)$ along π is x. Let $\gamma(x)$ be the subpath of π from p to $p(x)$. Furthermore, let π_i be the i-path to p, and let $\pi'_i(x)$ be the locally shortest path homotopic to the concatenation of π_i and $\gamma(x)$. The length of $\pi'_i(x)$ is denoted by $l_i(x)$ for $0 \leq x \leq 1$. Note that $l_i(0) < l_j(0)$ for $i < j$. If $l_i(x) \neq l_j(x)$ for all $0 \leq x \leq 1$ and $i \leq k < j$, then it is clear that $H_k(p) \oplus \pi = H_k(p')$. For the sake of contradiction, let x^* be the smallest x such that $l_i(x^*) = l_j(x^*)$ for some $i \leq k < j$. Let r be the number of graphs that pass below this intersection. If $r = k - 1$, then $p(x^*)$ is on a k-wall, which is a contradiction. If $r < k - 1$, then there must be an $m \leq k$ such that $l_m(x^*) > l_j(x^*)$. But that means that $l_m(x) = l_j(x)$ for some $x < x^*$, contradicting the choice of x^*. Similarly, if $r > k - 1$, then there must be an $m > k$ such that $l_m(x^*) < l_i(x^*)$. But that means that $l_m(x) = l_i(x)$ for some $x < x^*$, again contradicting the choice of x^*. □

Lemma 4.2. *The cells of the $(\leq k)$-SPM are simply connected.*

Proof. For the sake of contradiction, assume there is a cell of the $(\leq k)$-SPM that is not simply connected. Let C be a cycle in this cell that is not contractible. If C contains only k-walls in its interior, then there must be a triple point with an angle larger than 180 degrees, which is not possible (a triple point is a Voronoi vertex of an additively weighted Voronoi diagram). Hence there must be an obstacle ω inside C. Let $p \in C$ and let the largest winding number of any path in $H_k(p)$ with respect to ω be r. By Lemma 4.1 we have $H_k(p) \oplus C = H_k(p)$, where C is followed in counterclockwise direction. However, $H_k(p) \oplus C$ must contain a path with winding number $r + 1$. This is a contradiction. □

Lemma 4.3. *The number of faces, walls, and triple points of the $(\leq k)$-SPM is $O(k^2 h)$.*

Proof. We express the recurrence relations and the initial values using generating functions, which are formal power series with the sequence values as coefficients [10]. In general, for a sequence of values g_i, the generating function $g(z)$ is

$$g(z) = \sum_{i \geq 0} g_i z^i.$$

For our sequences, we have

$$\begin{aligned}
F(z) &= zB(z) - 2zW(z) + z \\
B(z) &= z\,(2B(z) - 3W(z) - F(z)) + V(z) + zh \\
V(z) &\leq z\,(2B(z) - 3W(z) - 2F(z)) + z(h-1) \\
W(z) &= zV(z)
\end{aligned}$$

Note that the constant term is zero, because we assume $F_0 = V_0 = B_0 = W_0 = 0$.

For convenience we will leave the "z" argument of the functions implicit during our manipulations. We can immediately eliminate the function $W = zV$:

$$\begin{aligned}
F &= zB - 2z^2V + z \\
B &= z(2B - 3zV - F) + V + zh \\
V &\leq z(2B - 3zV - 2F) + z(h-1)
\end{aligned}$$

Next we substitute $F = zB - 2z^2V + z$ into the last two relations to obtain

$$\begin{aligned}
B &= z(2B - 3zV - (zB - 2z^2V + z)) + V + zh \\
V &\leq z(2B - 3zV - 2(zB - 2z^2V + z)) + z(h-1)
\end{aligned}$$

or, combining terms,

$$\begin{aligned}
(1 - 2z + z^2)B &= (1 - 3z^2 + 2z^3)V + z(h-z) \\
(1 + 3z^2 - 4z^3)V &\leq (2z - 2z^2)B - 2z^2 + z(h-1)
\end{aligned}$$

Substituting

$$B = V\frac{(1 - 3z^2 + 2z^3)}{(1-z)^2} + \frac{z(h-z)}{(1-z)^2}$$

into the inequality for V, we obtain

$$\begin{aligned}
(1 + 3z^2 - 4z^3)V &\leq V\,\frac{2z(1-z)(1-3z^2+2z^3)}{(1-z)^2} \\
&\qquad + \frac{2z^2(1-z)(h-z)}{(1-z)^2} - 2z^2 + z(h-1) \\
&= 2z(1 + z - 2z^2)V + \frac{2z^2(h-z)}{1-z} - 2z^2 + z(h-1)
\end{aligned}$$

Rearranging terms and simplifying, we obtain

$$V \leq \frac{z(1+z)(h-1)}{(1-z)^3}.$$

Recall that $(1-z)^{-3} = \sum_{i\geq 0} \binom{i+2}{2} z^i$, and hence

$$\begin{aligned}
V &\leq \frac{z(1+z)(h-1)}{(1-z)^3} \\
&= \sum_{i\geq 1} z^i(h-1)\left[\binom{i+1}{2} + \binom{i}{2}\right] \\
&= \sum_{i\geq 0} z^i(h-1)i^2.
\end{aligned}$$

Returning from the domain of generating functions to our original recurrence relations, we have

$$V_i \leq (h-1)i^2,$$

which immediately implies

$$W_i = V_{i-1} \leq (h-1)(i-1)^2.$$

Solving for $B(z)$ instead of $V(z)$ gives

$$B_i \leq (h-1)(3i^2 - 3i + 2) + 1.$$

Finally, using $F_i = B_{i-1} - 2W_{i-1} \leq B_{i-1}$, we get

$$F_i \leq (h-1)(3i^2 - 9i + 8) + 1.$$

□

Theorem 4.4. *The k-SPM of a polygonal domain with n vertices and h holes has complexity $O(k^2h + kn)$.*

Proof. We have already argued in the main body of the paper that the k-SPM has $O(k^2h)$ k-walls (and $(k-1)$-walls) and $O(kn)$ k-windows. For the sake of completeness, we finally need to argue that k-walls, $(k-1)$-walls, and k-windows cannot cross. As mentioned before, there is no k-path to a point that is on a $(k-1)$-wall, and hence $(k-1)$-walls and k-walls cannot cross. Furthermore, the k-path to a point on a k-window is unique and follows the k-window in some direction. As a result, k-windows behave like k-paths, and a k-window turns into a $(k+1)$-window as it crosses a k-wall. Thus, a k-window cannot cross a k-wall. Similarly, a k-window cannot cross a $(k-1)$-wall, since the window would be a $(k-1)$-window on the other side of the crossing. Hence, the complexity of the k-SPM is $O(k^2h + kn)$. □